Trinity College Dublin, School of Mathematics

# MA2342 - Homework 7 - Solutions

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with some comments and discussion added by the lecturer

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Consider the system described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\left(\frac{\partial \phi}{\partial t}\right)^2 - c^2\left(\frac{\partial \phi}{\partial x}\right)^2\right) - V(\phi)\,, \tag{0.1}$$

where $c$ is some constant. $\phi$ is a function of $x$ and $t$.

## Problem 1

### Question

Find the energy density and momentum density. Write integrals that define energy and momentum of the system.

### Answer

Notation :

$$\frac{\partial \phi}{\partial t} = \phi_t, \qquad \frac{\partial \phi}{\partial x} = \phi_x.$$

- Momentum density : $\mathcal{P} = \phi_t\,\phi_x$
- Energy density : $\mathcal{H} = \frac{1}{2}\left(\phi_t^2 + c^2\phi_x^2\right) + V(\phi)$
- Momentum: $P = \int \phi_t\phi_x\,\mathrm{d}x$
- Energy : $E = \int \frac{1}{2}\left(\phi_t^2 + c^2\phi_x^2\right) + V(\phi)\,\mathrm{d}x$

### Derivation

We will interchangeably use the following notations:

$$\frac{\partial\phi}{\partial t} \equiv \phi_t \equiv \partial_t\phi, \qquad \frac{\partial\phi}{\partial x} \equiv \phi_x \equiv \partial_x\phi.$$

Since $\phi$ is a scalar field, $\phi_t$ and $\phi_x$ cannot be misunderstood as the $x$ component of a vector.

*Comment: In the literature on special/general relativity, to avoid confusion, a notation with comma, like $\phi_{,\nu}$, is used to denote partial derivatives. Then it can be naturally applied to the derivatives of a vector, for instance $\partial_\mu A_\nu \equiv A_{\mu,\nu}$. For this notes, I leave notations of the author. It is fully legitimate in the context of the exercise.*

On the lecture we introduced stress-energy tensor

$$T_\mu{}^\nu = \partial_\mu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \delta_\mu{}^\nu\mathcal{L}\,. \tag{0.2}$$

$T_0{}^0$ is interpreted as the energy density, and its actually coincides with the Hamiltonian density:

$$\begin{aligned} T_0{}^0 \equiv \mathcal{H} &= \phi_t\frac{\partial\mathcal{L}}{\partial\phi_t} - \mathcal{L} \\ &= \phi_t^2 - \frac{1}{2}\left(\phi_t^2 - c^2\phi_x^2\right) + V(\phi) \\ &= \frac{1}{2}\left(\phi_t^2 + c^2\phi_x^2\right) + V(\phi) \end{aligned}$$

Therefore, the energy is given by

$$H = \int \frac{1}{2}\left(\phi_t^2 + c^2\phi_x^2\right) + V(\phi)\,\mathrm{d}x$$

$T_1{}^0$ is interpreted as the momentum density.

*You should not confuse the momentum of the system (which is conserved quantity following from the translation invariance of the action) with the conjugate to $\phi$ field $\pi(x) = \frac{\partial\mathcal{L}}{\partial\phi_t} = \phi_t$*

The momentum density is given by

$$T_1{}^0 = \mathcal{P} = \phi_x\frac{\partial\mathcal{L}}{\partial\phi_t} = \phi_t\,\phi_x$$

and the momentum is given by $P = \int dx\,\mathcal{P}$.

## Problem 2

### Question

Write down equations of motion.

### Answer

The equation of motion is

$$\phi_{tt} - c^2\phi_{xx} = -\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}$$

### Derivation

The Euler-Lagrange equation for continuous system reads

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \phi_x} + \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{\partial \mathcal{L}}{\partial \phi}$$

In our case, we have that

$$\frac{\partial \mathcal{L}}{\partial \phi_x} = -c^2\phi_x, \qquad \frac{\partial \mathcal{L}}{\partial \phi_t} = \phi_t$$

and therefore the equation of motion is

$$\phi_{tt} - c^2\phi_{xx} = -\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}$$

## Problem 3

### Question

Prove that the system is relativistic invariant if $c$ is interpreted as a speed of light. This means: consider a Lorentz boost transformation $(ct') = \gamma(ct) - \beta\,\gamma\,x$, $x' = \gamma\,x - \beta\,\gamma(ct)$ and show that a) the Lagrangian looks exactly the same in new (primed) coordinates as in the original coordinates and b) equations of motion look exactly the same.

### Proof

We know that

$$\begin{aligned}\frac{\partial}{\partial ct} &= \frac{\partial ct'}{\partial ct}\frac{\partial}{\partial ct'} + \frac{\partial x'}{\partial ct}\frac{\partial}{\partial x'}\\ \frac{\partial}{\partial x} &= \frac{\partial ct'}{\partial x}\frac{\partial}{\partial ct'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'},\end{aligned}$$

and we have that

$$\frac{\partial ct'}{\partial ct} = \gamma \qquad \frac{\partial ct'}{\partial x} = -\beta\gamma$$

$$\frac{\partial x'}{\partial ct} = -\beta\gamma \qquad \frac{\partial x'}{\partial x} = \gamma$$

From this, we get that

$$\begin{aligned}\frac{1}{c^2}\mathcal{L} &= \frac{1}{2}\left((\gamma\phi_{ct'} - \beta\gamma\phi_{x'})^2 - (\gamma\phi_{x'} - \beta\gamma\phi_{ct'})^2\right) - \frac{V(\phi)}{c^2}\\ &= \frac{1}{2}\left((\gamma^2 - \gamma^2\beta^2)\phi_{ct'}^2 - 2\beta\gamma^2\phi_{x'}\phi_{ct'} + 2\beta\gamma^2\phi_{x'}\phi_{ct'} - (\gamma^2 - \gamma^2\beta^2)\phi_{x'}^2\right) - \frac{V(\phi)}{c^2}\end{aligned}$$

We then use the fact that $\gamma^2 - \gamma^2\beta^2 = 1$ to rewrite this equation as

$$\begin{aligned}\frac{1}{c^2}\mathcal{L} &= \frac{1}{2}\left(\phi_{ct'}^2 - \phi_{x'}^2\right) - \frac{V(\phi)}{c^2}\\ \Rightarrow \mathcal{L} &= \frac{1}{2}\left(\phi_{t'}^2 - c^2\phi_{x'}^2\right) - V(\phi)\end{aligned}$$

This means that the Lagrangian density is a Lorentz invariant, since $V(\phi)$ is a scalar and is therefore itself invariant under Lorentz boost.
For the equation of motion, we have that

$$\frac{\partial^2}{\partial(ct)^2} = \frac{\partial}{\partial ct}\left(\gamma\frac{\partial}{\partial ct'} - \beta\gamma\frac{\partial}{\partial x'}\right)$$
$$= \gamma^2\frac{\partial^2}{\partial(ct')^2} - 2\beta\gamma^2\frac{\partial^2}{\partial(ct')\partial x'} + \beta^2\gamma^2\frac{\partial^2}{\partial x'^2}$$

Similarly, we have that

$$\frac{\partial^2}{\partial(x)^2} = \frac{\partial}{\partial x}\left(\gamma\frac{\partial}{\partial x'} - \beta\gamma\frac{\partial}{\partial ct'}\right)$$
$$= \gamma^2\frac{\partial^2}{\partial x'^2} - 2\beta\gamma^2\frac{\partial^2}{\partial(ct')\partial x'} + \beta^2\gamma^2\frac{\partial^2}{\partial(ct')^2}.$$

From linearity of derivatives, dividing the equation of motion accross by $c^2$ and substituting the last two expressions we had, we see that the cross derivatives cancel out and we get that

$$\phi_{(ct)(ct)} - \phi_{xx} + \frac{1}{c^2}\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = \gamma^2(1-\beta^2)\phi_{(ct')(ct')} - \gamma^2(1-\beta^2)\phi_{x'x'} + \frac{1}{c^2}\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}$$

We multiply back by $c^2$ and use the fact that $\gamma^2(1-\beta^2) = 1$ to recover the equation of motion in the new variables,

$$\phi_{t't'} - c^2\phi_{x'x'} = -\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi},$$

where, again, $\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}$ is invariant under the Lorentz boost because it is a scalar. This has the same form as the original equation, and therefore the equation of motion is invariant under Lorentz boost. □

## Problem 4

### Question

For $V = \frac{c^2}{2}\frac{1}{(1+\phi^2)^2}$, find a solution of equations of motion if it is known that this solution does not depend on time and it has the following boundary condition: $\phi = \pm\infty$ and $\frac{\partial\phi}{\partial x} = 0$ at $x = \pm\infty$.
Your final answer should be that this solution satisfies $\phi + \frac{1}{3}\phi^3 = x - x_0$. Do not solve this cubic equation explicitly (possible, but useless for our goals).

### Solution

Let us first observe that if $\phi$ has no dependance on $t$, the equation of motion reduces to

$$-c^2\phi_{xx} = -\frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{0.3}$$

We will study this using the first integral of motion, in analogy how it is done in Lagrangian mechanics of one degree of freedom/Newtonian equations of motion. Define the quantity

$$K = \frac{c^2}{2}\phi_x^2 - V(\phi). \tag{0.4}$$

Check now that this is the first integral of motion (i.e. independent of $x$):

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}x} &= \frac{c^2}{2}\phi_x\,\phi_{xx} - \frac{\mathrm{d}V}{\mathrm{d}\phi}\phi_x;\\ &= \phi_x\left(c^2\phi_{xx} - \frac{\mathrm{d}V}{\mathrm{d}\phi}\right).\end{aligned}$$

By the equation of motion (0.3), the right term is 0, so $K$ as defined is indeed independent of $x$, and, therefore, constant. Let us now study its asymptotic behavior as $x \to \infty$. Since it is constant, that behavior will be the same on all space. Since $\phi \to \infty$ and $\phi_x \to 0$ when $x \to \infty$, we have that

$$V(\phi) = \frac{c^2}{2}\frac{1}{(1+\phi^2)^2} \to 0$$

and that

$$c^2\phi_x^2 \to 0.$$

This means that $K = 0$ on the whole space. Rewriting the equation for the energy and making $V$ explicit, this yields the differential equation

$$-\frac{c^2}{2}\phi_x^2 = -\frac{c^2}{2}\frac{1}{(1+\phi^2)^2}, \tag{0.5}$$

which we can rewrite as, after taking the square root on both sides

$$(1+\phi^2)\,\mathrm{d}\phi = \pm\,\mathrm{d}x.$$

Integrating on both sides, yields

$$\phi + \frac{\phi^3}{3} = \pm(x - x_0),$$

Using the fact that $\phi$ is positive at infinity we chose the sign " + " in the equation above. This is the desired implicit formulation for $\phi$. □

## Problem 5

### Question

For the solution from the previous exercise, depict qualitatively the energy density as a function of $x$. Precise how energy density behaves near $x = x_0$ and $x = \pm\infty$ (i.e. provide the Taylor expansion of the energy density as a function of $x$ at these points, up to the first order that depends on $x$.)

### Answer

- At $x = x_0$, $\frac{\mathcal{H}(x)}{c^2} = 1 - 2(x-x_0)^2 + \mathcal{O}((x-x_0)^4)$
- At $x = \pm\infty$, $\frac{\mathcal{H}(x)}{c^2} = \frac{1}{(3\,x)^{4/3}} + \mathcal{O}(x^{-2})$

### Derivation

To describe the behavior of the energy density with respect to $x$, let us first make a few observations.
First, from the (0.4) and knowledge that $K = 0$ one has that $\frac{c^2}{2}\phi_x^2 = V(\phi)$. This allows us to express $\phi_x$ as a function of $\phi$. Hence expression for the energy density will simplify to (remember that $\phi$ is time independent) that

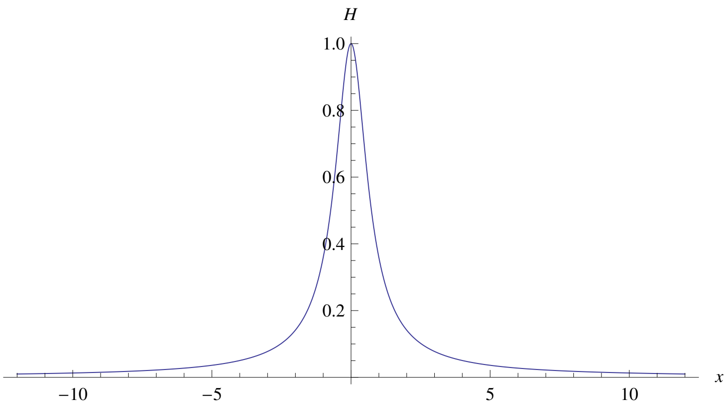
$$\mathcal{H}(\phi) = \frac{c^2}{2}\phi_x^2 + V(\phi) = 2\,V(\phi) = \frac{c^2}{(1+\phi^2)^2}\,. \tag{0.6}$$

On the other hand, we have

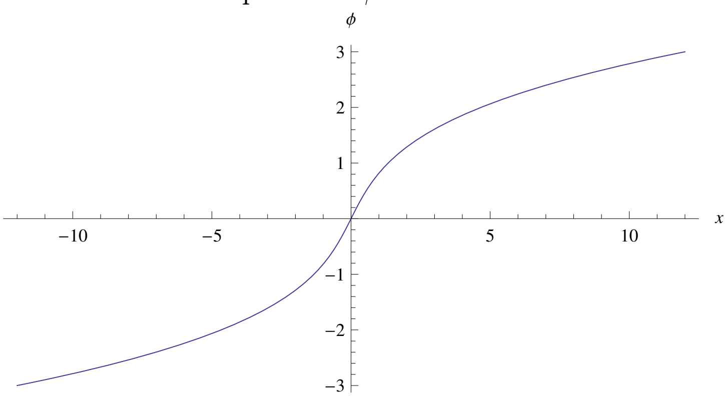
$$x(\phi) - x_0 = \phi + \frac{\phi^3}{3}, \tag{0.7}$$

The last two equations give us a parametric realisation of $H$ as a function of $\phi$, and can be explicitly plotted. Obviously, $x_0$ only shifts the whole picture along abscissa and $c$ is the overall normalisation. Hence, for the purpose of plotting we put $c = 1$ and $x_0 = 0$.
The plot can be constructed in principle qualitatively by hands, by analysing (0.6) and (0.7). We give the plot generated in *Mathematica*, and then perform its analysis analytically. The commands below generate the plot

```
H[\[Phi]_] = 1/(1 + \[Phi]^2)^2; x[\[Phi]_] = \[Phi] + \[Phi]^3/3;
ParametricPlot[{x[\[Phi]], H[\[Phi]]}, {\[Phi], -3, 3},
 AspectRatio -> .5, AxesLabel -> {x, H}]
```




The plot has a clear peak at 0 (generally, $x_0$). This is the place where the energy of the excitation is concentrated.
It is instructive to plot also $\phi$ versus $x$:



The field $\phi$ changes from $-\infty$ to $+\infty$ along abscissa. The fastest change/transition happens again around 0. This is the so called kink solution. Note that $-\infty$ and $+\infty$ are two minima of the potential $V(\phi)$. $\phi$ propagates between them.

**Analysis:** We have that $x(\phi)$ is bijective from $\mathbb{R} \to \mathbb{R}$ (this is seen from the fact that $x_\phi = 1+\phi^2$ is positive for all $\phi$ and that when $x \to \pm\infty$, $\phi \to \pm\infty$), and that it is always increasing, without extremas, and with an inflexion point at $x = x_0$. This means that, in terms of behavior, we can replace $\phi$ with $x - x_0$ in the definition for $\mathcal{H}$, and the only discrepency will be in terms of the speed of decay at infinity, and general decaying behavior.
From this analysis, we can deduce that $\mathcal{H}$ is symmetric with respect to $x = x_0$ (if $x_0 = 0$, it is simply even). We can also deduce that it attains a maximum at $x = x_0$ (where $\phi = 0$), and that it decays to 0 at infinity.
Let us now find the Taylor expansion around $x = x_0$. For this, lets us first study the Taylor expansion of $\phi$ around $x = x_0$. Observe that $\phi(x_0) = 0$, and from (0.5) $\phi_x(x_0) = 1$, because $\phi(x_0) = 0$. The second derivative $\phi_{xx}(x_0) = 0$ as can be read from $c^2\phi_{xx} = \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0$ at $x_0 = 0$. Therefore, the Taylor expansion of $\phi(x)$ around $x = x_0$ is

$$\phi(x) = (x - x_0) + \mathcal{O}((x - x_0)^3).$$

The Taylor expansion of $\mathcal{H}$ with respect to $\phi$ around $\phi = 0$ is

$$\mathcal{H}(\phi) = c^2 - 2c^2\phi^2 + \mathcal{O}(\phi^4).$$

Now we replace $\phi$ with its own Taylor expansion in $(x - x_0)$ to get

$$\mathcal{H}(x) = c^2 - 2c^2(x - x_0)^2 + \mathcal{O}((x - x_0)^4).$$

To study the behavior at $x \to \pm\infty$, we again put $x_0 = 0$ for simplicity. Equation (0.7) tells us that if $x$ is large, $\phi$ is large also. But this means that $x \sim \phi^3 \gg \phi$. Hence we can rewrite (0.7) in a way suitable for large $x$ Taylor expansion:

$$\phi = (3(x - \phi))^{1/3} = (3x)^{1/3} \times \left(1 - \frac{\phi}{x}\right)^{1/3} = (3x)^{1/3}(1 + O(x^{-2/3})) \tag{0.8}$$

where we used that $\phi \sim x^{1/3}$, so $\frac{\phi}{x} \sim x^{-2/3}$.
It remains to expand $\mathcal{H}$ at large $\phi$:

$$\mathcal{H} = \frac{c^2}{(1+\phi^2)^2} = \frac{c^2}{\phi^4}\frac{1}{(1+\frac{1}{\phi^2})^2} = \frac{c^2}{\phi^4} + O(\phi^{-6}) \tag{0.9}$$

and then substitute expansion of $\phi$, (0.8) to get

$$\mathcal{H} = \frac{c^2}{(3x)^{4/3}}(1 + O(x^{-2/3})) + O(x^{-2}) = \frac{c^2}{(3x)^{4/3}} + O(x^{-2})\,. \tag{0.10}$$

If we replace now $x \to x - x_0$ to consider generic case, conclusion will not change as is easy to check.

**Energy and momentum of the kink solution.** This is a "bonus" content which will allow you to better feel physics of the solution. It was not a part of the HW task.
We can compute the energy of the kink (recall that $\phi_x^2 = \frac{2V}{c^2}$):

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} \mathcal{H} dx = \int_{-\infty}^{+\infty} (2V) dx = \int_{-\infty}^{+\infty} (2V) \frac{dx}{d\phi} dx = \int_{-\infty}^{+\infty} (2V) \sqrt{\frac{c^2}{2V}} d\phi \\ &= c^2 \int_{-\infty}^{+\infty} \frac{1}{(1+\phi^2)} d\phi = \pi\, c^2 \,. \end{aligned} \tag{0.11}$$

The momentum of the solution is $P = \int \phi_t\, \phi_x = 0$ which simply means that energy is not transferred in space. Indeed, solution does not move.

Since the theory is relativistically invariant, we can make sense of Einsten formulae. For instance, from $E = m\, c^2$ we conclude that the kink solution describes an excitation of mass $m = \pi$.

Relativistic invariance also means that we can make a boost of our solution. Practically, $\phi(\gamma(x - v\, t))$ will be also a solution of the equations of motion. In principle it was checked in exercise 3, but let us do it explicitly here. The check is $\partial_x^2 \phi(\gamma(x - v\, t)) = \gamma^2 \frac{1}{c^2} \frac{\mathrm{d}V}{\mathrm{d}\phi}$ and $\partial_t^2 \phi(\gamma(x - \beta\, t)) = v^2 \gamma^2 \frac{1}{c^2} \frac{\mathrm{d}V}{\mathrm{d}\phi} = \beta^2 \gamma^2 \frac{\mathrm{d}V}{\mathrm{d}\phi}$. Buy substituting these expressions to the equation of motion, it is easy to see that it is satisfied.
In other words, we know that $\phi$ implicitly defined by

$$\phi(x,t) + \frac{\phi(x,t)^3}{3} = \gamma((x - x_0) - v\, t) \tag{0.12}$$

is also a solution of equations of motion. For this solution, the energy density would move with velocity $v$. Note that $\gamma$ is real only for $|v| < c$.
We can compute the energy for this solution. In $\int_{-\infty}^{+\infty} \mathcal{H}(\phi(x,t)) dx$ we make a change of variables $x \to \gamma((x - x_0) - v\, t)$, so $dx \to \gamma dx$. After this change, the integrals in (0.11) are exactly the same, but with the common prefactor $\gamma$. So we get $E = \gamma\, \pi\, c^2$, i.e. exactly the relativistic energy of a massive moving object.
The momemntum now will be non-zero: $\phi_t = -\beta\, \gamma \sqrt{2V} = -\beta\, \gamma\, c \frac{1}{1+\phi^2}$, so

$$P = \int \phi_t \phi_x \, dx = \int \phi_t d\phi = \int (-\beta \gamma c \frac{1}{1+\phi^2}) d\phi = -\gamma\, \pi\, v \,. \tag{0.13}$$

It is, up to a sign, the relativistic momentum of a moving object. Why is the sign wrong? We recall that $P$ is actually $P_1$, a space component of $P_\mu = \int T_\mu{}^0 dx$. Since the theory is relativistic, we can apply all the relevant formalism, in particular of raising indices. Raising space component changes the sign, so we get the correct 2-momentum:

$$P^\mu = \{E/c, P^1\} = \{\gamma\, \pi\, c, \gamma\, \pi\, v\} \,. \tag{0.14}$$