HW7 ①

Solutions

1. $\beta = \dfrac{\beta' + \beta''}{1+\beta'\beta''}$

β'' – velocity of the particle in the ref. frame O'

β' – velocity of ref. frame O' with respect to ref. frame O

β – velocity of the particle in the ref. frame O

Approach 1: $\left|\dfrac{\beta'+\beta''}{1+\beta'\beta''}\right| * 1 \iff |\beta'+\beta''| * |1+\beta'\beta''|$

$\uparrow$ $*$ is either $<$ or $>$ have to decide

Take square:

$$(\beta'+\beta'')^2 * (1+\beta'\beta'')^2$$

$$(\beta')^2 + \cancel{2\beta'\beta''} + (\beta'')^2 * 1 + \cancel{2\beta'\beta''} + (\beta'\beta'')^2$$

$$(\beta')^2 + (\beta'')^2 * 1 + (\beta'\beta'')^2$$

Take square (again):

$$\left((\beta')^2 + (\beta'')^2\right)^2 * \left(1 + (\beta'\beta'')^2\right)^2$$

$$(\beta')^4 + (\beta'')^4 + \cancel{2(\beta'\beta'')^2} * 1 + (\beta'\beta'')^4 + \cancel{2(\beta'\beta'')^2}$$

$$(\beta')^4 + (\beta'')^4 * 1 + (\beta'\beta'')^4$$

$$\boxed{(\beta')^{2^n} + (\beta'')^{2^n} * 1 + (\beta'\beta'')^{2^n}, \quad n = 1, 2, 3, \dots}$$

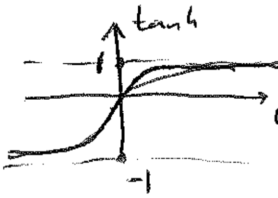
$*$ is the same for all equalities written.

Since $|\beta'|<1$, $|\beta''|<1$ and hence $|\beta'\beta''|<1$, $\exists\, n$: $(\beta')^{2^n} < 1/2$, $(\beta'')^{2^n} < 1/2$

For such n it is obvious that $* = $ "$<$"

So $\left|\dfrac{\beta'+\beta''}{1+\beta'\beta''}\right| < 1$

Approach 2: Parameterise $\beta' = \tanh\theta'$, $\beta'' = \tanh\theta''$



for $\forall\, \theta \in \mathbb{R}$ $|\tanh\theta| < 1$

$\Downarrow$

Now use trigonometry to get:

$$\frac{\tanh\theta' + \tanh\theta''}{1 + \tanh\theta' \tanh\theta''} = \tanh(\theta' + \theta'') = \beta \implies |\beta| < 1.$$

$\boxed{1 - \text{continued}}$ If $\beta'' = 1$ then $\beta = 1$, independently of β'.

Physical interpretation: If an object moves with the speed of light $(\beta''=1)$, it moves with the speed of light in any ref.-frame $(\beta = 1)$.

$\boxed{2}$ $\beta = \tanh\theta$ $\quad \gamma = \cosh\theta$ (note, $\gamma > 1$ always)

$\beta\gamma = \sinh\theta$

$$\beta = \frac{\beta' + \beta''}{1+\beta'\beta''} \iff \theta = \theta' + \theta''$$

$\boxed{3}$ $$E(\beta) = \frac{1}{\sqrt{1-\beta^2}} mc^2 = \gamma mc^2$$

$$p^\mu = (E/c, \vec{p}) = \gamma m(c, \vec{v}).$$

4-momentum

Note: $p^\mu p_\mu = p^0 p^0 - \vec{p}\cdot\vec{p} =$
$= (\gamma m)^2 (c^2 - \vec{v}^2) =$
$= m^2 c^2 \frac{1-\vec{\beta}^2}{1-\vec{\beta}^2} = m^2 c^2$

It transform in the same way as coordinates do (contra-variantly)

Proof: $$(p^\mu)' = \gamma' m(c, v') = (E'/c, p')$$

Let α - velocity of O' with respect to O. Then $\beta = \frac{\alpha + \beta'}{1+\alpha\beta'}$

$$\beta' = \frac{\beta - \alpha}{1-\beta\alpha}$$

$$\gamma' = \frac{1}{\sqrt{1-(\beta')^2}} = \frac{1}{\sqrt{1-\left(\frac{\beta-\alpha}{1-\beta\alpha}\right)^2}} = \frac{1-\beta\alpha}{\sqrt{1-\cancel{2\beta\alpha}+\beta^2\alpha^2-\beta^2-\alpha^2}} =$$

$$= \frac{1-\beta\alpha}{\sqrt{1-\beta^2}\sqrt{1-\alpha^2}}$$

$$(p^\mu)' = \frac{1-\beta\alpha}{\sqrt{1-\beta^2}\sqrt{1-\alpha^2}} m\left(c, c\cdot\frac{\beta-\alpha}{1-\beta\alpha}\right) = \gamma_\beta m\left(\gamma_\alpha c - \gamma_\alpha \beta\alpha c, \gamma_\alpha c\cdot\beta - \gamma_\alpha \cdot c\cdot\alpha\right) =$$

[3] continued

$$= \left(\gamma_\alpha \frac{E_\beta}{c} - p_\beta \cdot \gamma_\alpha \alpha,\ \gamma_\alpha \cdot p_\beta - \frac{E_\beta}{c}\gamma_\alpha \cdot \alpha\right) \Rightarrow$$

$$\Rightarrow \begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & -\alpha\gamma \\ -\alpha\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix} \quad \text{(as required)}$$

$\boxed{4}$ Substitute $E = Ac^2 \cosh\theta$, $p = Ac \sinh\theta$ to $(E/c)^2 - \vec{p}^2 = m^2c^2$ and conclude that $A = m$.

change of θ: $\theta \rightarrow \theta + \theta'$ (see question 2)

$\boxed{5}$ ~~It is enough to do 1+1 dim case as we did not discuss generic addition of velocities formula in 1+3 dim~~

~~We know that $((E_i/c, p_i)$ th~~

$\boxed{5}$ We know that $(E_i/c, p_i)$ transforms as 4-vector under ~~change~~ Lorentz transformations:

$$\begin{pmatrix} E_i'/c \\ \vec{p}_i' \end{pmatrix} = \underset{\substack{\uparrow \\ \text{Lorentz matrix } (4\times 4)}}{L} \begin{pmatrix} E_i/c \\ \vec{p}_i \end{pmatrix}$$

By linearity, the same is true for $E = \sum_i E_i/c$; $\vec{P} = \sum_i \vec{p}_i$:

$$\begin{pmatrix} E'/c \\ \vec{P}' \end{pmatrix} = L \begin{pmatrix} E/c \\ \vec{P} \end{pmatrix}$$

for $\frac{dL}{da} = 0$, one has

$$\frac{d}{da}\begin{pmatrix} E'/c \\ \vec{p}' \end{pmatrix} = \frac{d}{da}\left(L \cdot \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}\right) = L \underbrace{\frac{d}{da}\begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}}_{\substack{= 0 \\ \text{by condition} \\ \text{of the problem}}} = 0$$

6 If $(x^\mu)' = L^\mu{}_\nu x^\nu$, then

$$A'_{\mu\nu} = (L^{-1})^{\mu'}{}_\mu (L^{-1})^{\nu'}{}_\nu \, A_{\mu'\nu'}$$

Or in index-free notation:

$$\boxed{A' = (L^T)^{-1} \cdot A \cdot L^{-1}}$$

For $L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad L^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

So $A' = \begin{pmatrix} \gamma & \beta\gamma & - & - \\ \beta\gamma & \gamma & - & - \\ - & - & - & - \end{pmatrix} A \begin{pmatrix} \gamma & \beta\gamma & - & - \\ \beta\gamma & \gamma & - & - \\ - & - & - & - \end{pmatrix} =$

$$\cancel{= \Big(\gamma^2 (A_{00} + \beta A_{01} + \beta A_{10} + \beta^2 A_{11})}$$

$$(*) \quad = \begin{pmatrix} \gamma^2 (A_{00} + \beta A_{01} + \beta A_{10} + \beta^2 A_{11}) & \gamma^2 (A_{01} + \beta A_{00} + \beta A_{11} + \beta^2 A_{10}) & \gamma (A_{02} + \beta A_{12}) & \gamma (A_{03} + \beta A_{13}) \\ \gamma^2 (A_{10} + \beta A_{00} + \beta A_{11} + \beta^2 A_{01}) & \gamma^2 (A_{11} + \beta A_{10} + \beta A_{01} + \beta^2 A_{00}) & \gamma (A_{12} + \beta A_{02}) & \gamma (A_{13} + \beta A_{03}) \\ \gamma (A_{20} + \beta A_{21}) & \gamma (A_{21} + \beta A_{20}) & A_{22} & A_{23} \\ \gamma (A_{30} + \beta A_{31}) & \gamma (A_{31} + \beta A_{30}) & A_{32} & A_{33} \end{pmatrix}$$

7 You can use for instance explicit matrix (*) above (of course, there are easier ways). Put there $A_{\mu\nu} = \eta_{\mu\nu}$. You get:

$$\begin{pmatrix} \gamma^2 (1 - \beta^2) & \gamma^2 (\beta - \beta) & \gamma \cdot 0 & \gamma \cdot 0 \\ \gamma^2 (\beta - \beta) & \gamma^2 (\beta^2 - 1) & \gamma \cdot 0 & \gamma \cdot 0 \\ \gamma \cdot 0 & \gamma \cdot 0 & -1 & 0 \\ \gamma \cdot 0 & \gamma \cdot 0 & 0 & -1 \end{pmatrix} \underset{\gamma^2 = (1-\beta^2)^{-1}}{=} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

8 From $A' = (L^T)^{-1} \cdot A \cdot L^{-1}$

it follows $\det A' = \det\left((L^T)^{-1} \cdot A \cdot L^{-1}\right) = (\det L^{-1})^2 \cdot \det A$

since $\det L = \pm 1 \Rightarrow \det A' = \det A$

Conclusion: determinant of rank 2 covariant tensor is invariant under Lorentz transformations.

9 2 points are given for the following explanation (or similar)

in the reference frame of the rod, simultaneous blocking of ~~th~~ two beams are two space-time events A & B (see Fig on next page).

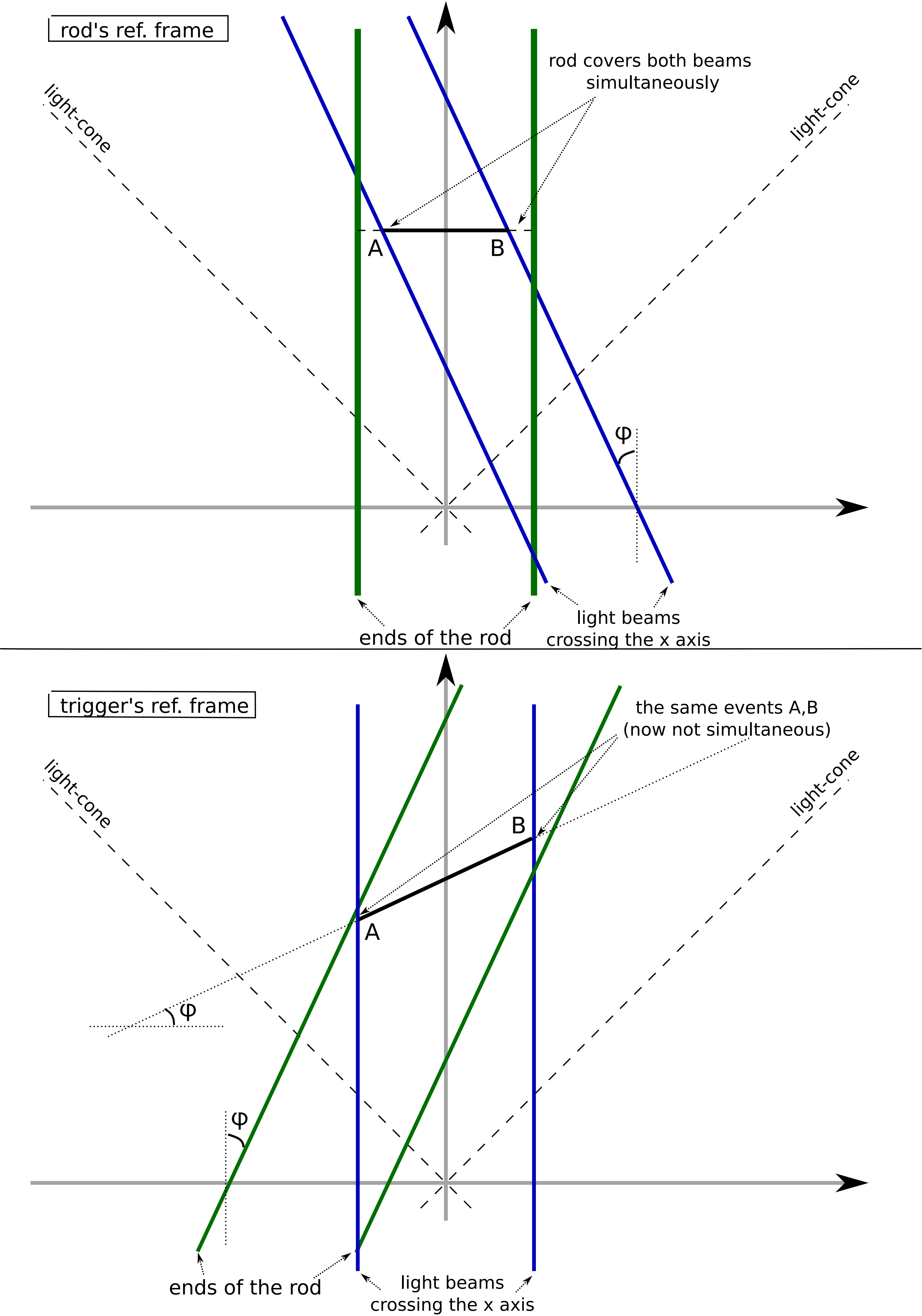
These two events are <u>not simultaneous</u> in the trigger's ref. frame.

Alarm will sound only when light beams are simultaneously blocked in trigger's frame. For events A & B this not the case $\Rightarrow$ <u>alarm will not sound</u>

Full score is only for clear explanation. In particular, if you have qualitatively wrong pictures, it will be penalised

+0.5 points if you draw picture with correct angles.

+0.5 points if you explain clearly why simultaneous blocking of ~~lig~~ beams in the <u>rod's frame</u> does not trigger alarm (~~I mean~~ ~~expl~~ i.e. discuss how absence of light propagates up to the trigger).


rod's ref. frame
rod covers both beams simultaneously
light-cone
light-cone
A
B
φ
ends of the rod
light beams crossing the x axis
trigger's ref. frame
the same events A,B (now not simultaneous)
light-cone
light-cone
B
A
φ
φ
ends of the rod
light beams crossing the x axis