 HW5 solutions

1. $dS_1 = p\,dq - P\,dQ - (\mathcal{H}-\mathcal{H}')dt$

$S_1(q,Q,t)$: $\boxed{p = \frac{\partial S_1}{\partial q}\Big|_{Q,t};\ P = -\frac{\partial S_1}{\partial Q}\Big|_{q,t};\ \mathcal{H}'-\mathcal{H} = \frac{\partial S_1}{\partial t}\Big|_{q,Q}}$

S_2, S_3, S_4 - are Legendre transforms of S_1 (but with respect to different variables/ pair of variables)

$S_2(q,P,t) = S_1 + PQ$

$dS_2 = dS_1 + P\,dQ + Q\,dP = p\,dq + Q\,dP - (\mathcal{H}-\mathcal{H}')dt$

$$\boxed{p = \frac{\partial S_2}{\partial q}\Big|_{P,t};\quad Q = +\frac{\partial S_2}{\partial P}\Big|_{q,t};\quad \mathcal{H}'-\mathcal{H} = \frac{\partial S_2}{\partial t}\Big|_{q,P}}$$

showing what is fixed is important (compare to a similar expression for S_1)

time is usually fixed by default. So if you do not mention that it is fixed, it is ok.

$S_3(p,Q,t) = S_1 - pq$

$dS_3 = -q\,dp - P\,dQ - (\mathcal{H}-\mathcal{H}')dt$

$$\boxed{q = -\frac{\partial S_3}{\partial p}\Big|_{Q,t};\quad P = -\frac{\partial S_3}{\partial Q}\Big|_{p,t};\quad (\mathcal{H}'-\mathcal{H}) = \frac{\partial S_3}{\partial t}\Big|_{p,Q}}$$

$S_4(p,P,t) = S_1 + PQ - pq$

$dS_4 = -q\,dp + Q\,dP - (\mathcal{H}-\mathcal{H}')dt$

$$\boxed{q = -\frac{\partial S_4}{\partial p}\Big|_{P,t};\quad Q = +\frac{\partial S_4}{\partial P}\Big|_{p,t};\quad (\mathcal{H}'-\mathcal{H}) = \frac{\partial S_4}{\partial t}\Big|_{p,P}}$$

2. (a)
$$q_i = -\frac{\partial S_3}{\partial p_i}\Big|_Q = Q^i + a^i$$
$$P_i = -\frac{\partial S_3}{\partial Q_i}\Big|_p = p_i$$
$$\Rightarrow \quad \vec{Q} = \vec{q} - \vec{a}, \qquad \vec{P} = \vec{p}$$
(meaning: translation by $-\vec{a}$)

(b)
$$p_i = \frac{\partial S_2}{\partial q_i}\Big|_P = P_i - \epsilon_{jik} a^j P_k = P_i + \epsilon_{ijk} a^j P_k$$

$$Q_i = \frac{\partial S_2}{\partial P_i}\Big|_q = q_i - \epsilon_{jki} a^j q_k = q_i - \epsilon_{ijk} a^j q_k$$

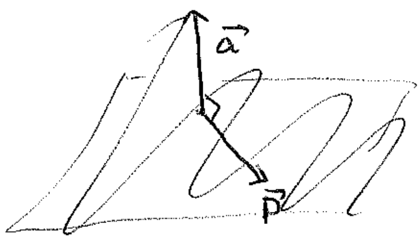
$$P_i = p_i - \epsilon_{ijk} a^j P_k \overset{\text{linear approx.}}{=} p_i - \epsilon_{ijk} a^j p_k$$

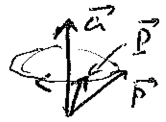
[explanation:
$P = p - a \times P =$
$= p - a \times (p - a \times P) =$
$= p - a\times p + \underbrace{a \times (a \times P)}_{\text{quadratic in } a, \text{ so neglect}}$]

$$Q_i = q_i - \epsilon_{ijk} a^j q_k$$

Or in vector form:

$$\vec{P} = \vec{p} + \vec{p} \times \vec{a} \qquad \text{(meaning: rotation}$$
$$\vec{Q} = \vec{q} + \vec{q} \times \vec{a}$$





If you write $\vec{a} = \tau \vec{n}$, then you can think about
$\vec{p} = \vec{P}(0)$
$\vec{P} = \vec{P}(\tau)$

$$\underbrace{\frac{d\vec{P}}{d\tau} = \vec{P} \times \vec{n}}$$
↑ standard dif. equation for rotation

(c)
$$q = -\frac{\partial S_3}{\partial p}\Big|_Q ; \quad P = -\frac{\partial S_3}{\partial Q}\Big|_p$$

~~Consider~~

$$q = Q + \tau \frac{\partial \mathcal{H}}{\partial p}\Big|_Q \qquad\Rightarrow\qquad Q = q - \tau \frac{\partial \mathcal{H}}{\partial p}\Big|_Q$$
$$P = p + \tau \frac{\partial \mathcal{H}}{\partial Q}\Big|_p \qquad\qquad P = p + \tau \frac{\partial \mathcal{H}}{\partial Q}\Big|_p$$

2c - Continued

Considering linear in τ approximation,

we can replace
$$\frac{\partial \mathcal{H}(p,Q)}{\partial p} \to \frac{\partial \mathcal{H}(p,q)}{\partial p}$$
$$\frac{\partial \mathcal{H}(p,Q)}{\partial Q} \to \frac{\partial \mathcal{H}(p,q)}{\partial q}$$

Thinking about $\begin{matrix} Q = q(\tau) \\ P = p(\tau) \end{matrix} \quad \left(\begin{matrix} q = q(0) \\ p = p(0) \end{matrix} \right)$, we get

a differential equations in the limit $\tau \to 0$

$$\left.\begin{aligned} \frac{dq}{d\tau} &= -\frac{\partial \mathcal{H}}{\partial p} \\ \frac{dp}{d\tau} &= \frac{\partial \mathcal{H}}{\partial q} \end{aligned}\right\} \Rightarrow (-\mathcal{H}) \text{ plays the role of hamiltonian}$$

and canonical transformation: solution of equs of motion.

2d) $S_1(q,Q) = \frac{m}{2}\frac{(Q-q)^2}{t_2-t_1}$

$$\left.\begin{aligned} p &= \left.\frac{\partial S_1}{\partial q}\right|_Q = \frac{m}{t_2-t_1}(q-Q) \\ P &= -\left.\frac{\partial S_1}{\partial Q}\right|_q = \frac{m}{t_2-t_1}(q-Q) \end{aligned}\right\}$$

~~$\Rightarrow P = p$~~ ~~...~~ situation is not clear.

$$\Rightarrow P = p$$
$$Q = -\frac{t_2-t_1}{m}p + q$$

$$\left.\begin{aligned} p &= P \\ q &= Q + \frac{t_2-t_1}{m}p \end{aligned}\right\} \Leftarrow \text{in this way interpretation is better.}$$

It is a free particle moving with momentum p, its position after $\Delta t = t_2 - t_1$.

3.

HW5 (4) solutions

Step 1: find solution of equations of motion. For a free particle it is just

$$q(t) = \alpha t + \beta$$

$$q(0) = q_i \; ; \; q(t_f) = q_f$$

$$\Downarrow$$

$$\beta = q_i \qquad \alpha = \frac{q_f - q_i}{t_f}$$

$$q(t) = \frac{q_f - q_i}{t_f} \cdot t + q_i$$

Step 2 substitute this solution into S^{cl}:

$$S(q_f, q_i, t_f) = \frac{m}{2} \int_0^{t_f} \dot{q}^2 dt = \frac{m}{2} \int_0^{t_f} \left(\frac{q_f - q_i}{t_f} \right)^2 dt = \frac{m}{2} \frac{(q_f - q_i)^2}{t_f}$$

Hence

$$S(q_f, q_i, t) = \frac{m}{2} \frac{(q_f - q_i)^2}{t}$$

Note that this a generating function from 2(d).

~~next page~~

4. Canonical transformation $X = X(x)$ has the property:

HW5 (4)

solutions (5)

$$\{X_\alpha, X_\beta\}_x = \omega_{\alpha\beta}$$

as discussed on the lecture.

This property is the necessary and sufficient condition for the transformation to be canonical.

In practice, we have to demand:

$$\{X, Y\} = 0 \qquad \{X, P_Y\} = 0$$
$$\{P_X, P_Y\} = 0 \qquad \{Y, P_X\} = 0$$
$$\{X, P_X\} = 1 \qquad \{Y, P_Y\} = 1$$

Result of computation of Poisson brackets:

$$\{X, Y\} = \frac{1-\beta^2}{\alpha^2} \qquad \{X, P_y\} = 0$$
$$\{P_X, P_y\} = \frac{\alpha^2}{4}(1-\beta^2) \qquad \{y, P_X\} = 0$$

$$\{X, P_X\} = \frac{1}{2}(1+\beta^2)$$
$$\{y, P_y\} = \frac{1}{2}(1+\beta^2)$$

From the last one: $\frac{1}{2}(1+\beta^2) = 1 \Rightarrow \beta^2 = 1 \Rightarrow \underline{\beta = \pm 1}$

This is enough to satisfy all the $\{\cdot,\cdot\}$ to be correct.

So: $\boxed{\beta = \pm 1, \ \alpha - \text{arbitrary}}$ ($\alpha \neq 0$ though)

5. From question 3: $S(q, Q, t) = \frac{m}{2}\frac{(Q-q)^2}{t}$

HW5 (6)
solutions

HJ equation $\frac{\partial S}{\partial t} + \mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) = 0$

$$\underline{\frac{\partial S}{\partial t} = -\frac{m}{2}\frac{(Q-q)^2}{t^2}}$$

$\mathcal{H}(q,p,t) = \frac{p^2}{2m}$ (for free particle) $\Rightarrow$ $\mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) =$

$$= \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 =$$

$$= \frac{1}{2m}\left(m\frac{q-Q}{t}\right)^2 =$$

$$= \underline{\frac{m}{2}\frac{(Q-q)^2}{t^2}}$$

So indeed HJ is satisfied.

6.

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 = 0; \qquad S = \overbrace{W(q,E)}^{\text{function only of } q} - \overbrace{Et}^{\text{function only on } t}$$

$$-E + \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 = 0 \Rightarrow E = const \quad \text{(does not depend on } t\text{)}$$

$$\frac{\partial W}{\partial q} = \sqrt{2mE} \Rightarrow W = \sqrt{2mE}\,(q - q_0) + f(E) \qquad (*)$$

$S(q,Q,t) = \frac{m}{2}\frac{(Q-q)^2}{t}$. For the sake of comparison we define Legendre transform with the following signs:

$$Leg.(S) = \sup_t (Et + S)$$

Hence $E = -\frac{\partial S}{\partial t} = \frac{m}{2}\frac{(Q-q)^2}{t^2} \Rightarrow t = \pm\sqrt{\frac{m}{2E}}(Q-q)$

$$Et + S = \left(t = \pm\sqrt{\frac{m}{2E}}(Q-q)\right) = \pm\sqrt{\frac{mE}{2}}(Q-q) + \pm\frac{m}{2}\frac{Q-q}{\sqrt{\frac{m}{2E}}} = \pm\sqrt{\frac{mE}{\cancel{2}}}(Q-q)$$

with "–" choice we get (*).

7. $p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} + By \quad \Rightarrow \dot{x} = p_x - By$

$p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{y} - Bx \quad \Rightarrow \dot{y} = p_y + Bx$

$\mathcal{H}(x, y, p_x, p_y) = p_x \dot{x} + p_y \dot{y} - \mathcal{L} =$ (substitute all $\dot{x}, \dot{y}$) $=$

$$= p_x(p_x - By) + p_y(p_y + Bx) - \frac{1}{2}(p_x - By)^2 - \frac{1}{2}(p_y + Bx)^2 -$$

$$- B\left((p_x - By)y - (p_y + Bx)x\right) + \frac{1}{2}\omega^2(x^2+y^2) =$$

$$= \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - \cancel{B\cdot(p_x y - p_y x)} + \cancel{B(p_x y - p_y x)} -$$

$$-\frac{1}{2}B^2y^2 - \frac{1}{2}B^2x^2 - B(p_x y - p_y x) +$$

$$+ B^2(y^2 + x^2) + \frac{1}{2}\omega^2(x^2+y^2) =$$

$$\boxed{H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - B(p_x y - p_y x) + \frac{1}{2}(\omega^2 + B^2)(x^2+y^2).}$$

$\mathcal{L}_{\substack{first\\order}} = p_x \dot{x} + p_y \dot{y} - H$, H from the box above.

8. Hamiltonians are not invariant under time-dependent transformation, recall $\mathcal{H}' - \mathcal{H} = \frac{\partial S}{\partial t}$. But Lagrangians are invariant. This means that we can boldly substitute $\begin{matrix} x = x(X, Y, t) \\ p = p(P, Y, t) \end{matrix}$ into Lagrangian,

Note
$p_x \dot{x} + p_y \dot{y}$ is interpreted as $\vec{p} \cdot \dot{\vec{x}}$, where $\vec{p} \equiv (p_x, p_y)$, $\dot{\vec{x}} = (\dot{x}, \dot{y})$

If $\vec{x} = \mathcal{O}\vec{X}$, where $\mathcal{O} = \begin{pmatrix} \cos Bt & -\sin Bt \\ \sin Bt & \cos Bt \end{pmatrix}$,

then

$$\dot{\vec{x}} = \mathcal{O}\dot{\vec{X}} + \dot{\mathcal{O}}\vec{X} \quad ; \quad \dot{\mathcal{O}} = -B\begin{pmatrix} +\sin Bt & +\cos Bt \\ -\cos Bt & \sin Bt \end{pmatrix}$$

$$\vec{p}\cdot\dot{\vec{x}} = \vec{P}^{\overset{\text{transposed}}{T}} \mathcal{O}^T(\mathcal{O}\dot{\vec{X}} + \dot{\mathcal{O}}\vec{X}) = \vec{P}\dot{\vec{X}} + \vec{P}^T\mathcal{O}^T\dot{\mathcal{O}}\vec{X}$$

8. - continued

$$\mathcal{O}^T \dot{\mathcal{O}} = -B \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} \begin{pmatrix} \sin & \cos \\ -\cos & \sin \end{pmatrix} = -B \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\varepsilon}$$

Conclusion :

$$\vec{p}\,\dot{\vec{x}} = \vec{P}\,\dot{\vec{X}} - B\,(\vec{P}^T \varepsilon\, \vec{X})$$

Now rewrite H in vector form

$$H = \frac{1}{2}\vec{p}\cdot\vec{p} + \frac{1}{2}(\omega^2+B^2)\,\vec{x}\cdot\vec{x} - B\,(\vec{p}^T\varepsilon\cdot\vec{x})$$

Under rotations $\downarrow$

$$\frac{1}{2}\vec{P}\cdot\vec{P} + \frac{1}{2}(\omega^2+B^2)\,\vec{X}\cdot\vec{X} - B\,(\vec{P}^T\cdot \underbrace{\mathcal{O}^T\varepsilon\,\mathcal{O}}_{=\varepsilon}\,\vec{X})$$

Indeed: $\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} =$

$$= \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}\begin{pmatrix} \sin & \cos \\ -\cos & \sin \end{pmatrix} = \varepsilon$$

Hence, $\mathcal{L}$ after rotations of coordinates becomes:

$$\mathcal{L}_{\substack{\text{first}\\\text{order}}} = \vec{P}\,\dot{\vec{X}} \cancel{- B\,(\vec{P}^T\varepsilon\,\vec{X})} - \left(\frac{1}{2}\vec{P}^2 + \frac{1}{2}(\omega^2+B^2)\vec{X}^2 \cancel{- B(\vec{P}^T\varepsilon\,\vec{X})}\right)$$

Since $\mathcal{L} = p\dot{x} - \mathcal{H}$ always, we conclude that

$$\mathcal{H}' = \frac{1}{2}\vec{P}^2 + \frac{1}{2}(\omega^2+B^2)\,\vec{X}^2,$$ i.e 2 harmonic oscillators with frequency $\underline{\omega + B}$

9. Generic HJ is $\frac{\partial S}{\partial t} + \mathcal{H}' = 0.$

Since there is no explicit dependence on time, we represent $S = W - Et$ to get

$$\mathcal{H}'\left(q, \frac{\partial W}{\partial q}\right) = E$$

E is interpreted as total energy.

<u>9-continued</u> Since Hamiltonian is of the form:

$$\mathcal{H}' = \sum_{i=1}^{2} \left(\frac{1}{2} P_i^2 + \frac{1}{2}(\omega^2+B^2) X_i^2 \right),$$

i.e it splits into $\mathcal{H}' = \mathcal{H}_1(P_1, X_1) + \mathcal{H}_2(P_2, X_2)$,

we can write an ansatz:

$$W = W_1(X_1) + W_2(X_2)$$

and get equations from HJ (standard logic of separation of variables):

$$\frac{1}{2}\left(\frac{\partial W_i}{\partial X_i}\right)^2 + \frac{1}{2}(\omega^2+B^2) X_i^2 = E_i \,, \quad \text{for } i = 1,2$$

with $E_1 + E_2 = E$

Obviously, these 2 equations for $i = 1,2$ are equivalent.
We need to solve only one:

$$\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}(\omega^2+B^2) x^2 = E$$

This is HJ for Harmonic oscillator, it was explained during lecture

$$\underbrace{W = \int \sqrt{2E - (\omega^2+B^2)x^2}\, dx}_{(*)} \overset{\text{using hint}}{=} \frac{1}{2} x \sqrt{2E - (\omega^2+B^2) x^2} + \frac{E}{\sqrt{\omega^2+B^2}} \arcsin\left(\sqrt{\frac{\omega^2+B^2}{2E}}\, x \right)$$

$$W_{total} = \sum_{i=1}^{2} \left(\frac{1}{2} X_i \sqrt{2E_i - (\omega^2+B^2) X_i^2} + \frac{E}{\sqrt{\omega^2+B^2}} \arcsin\left(\sqrt{\frac{\omega^2+B^2}{2E}}\, X_i \right) \right)$$

$$\boxed{S = -(E_1+E_2)t + W_{total}}$$

~~10. We first use $P_i = \frac{\partial W}{\partial X_i}$. But from (*) it simply means $P_i = \sqrt{\ldots} \Rightarrow \frac{1}{2}(P_i^2 + (\omega^2+B^2) X_i^2) = E_i$~~

10. We can think about $S(X_1, X_2, E_1, E_2, t)$ as about generating function of canonical transformation $S_2(q, P, t)$ which

$$\underset{\substack{\uparrow \\ \text{this is } X_i}}{q}, \quad \underset{\substack{\uparrow \\ \text{this } E}}{P}$$

maps original Hamiltonian $\mathcal{H}'$ to the new hamiltonian $K=0$. Indeed $\underset{\substack{\uparrow \\ \text{new Hamiltonian}}}{K} - \underset{\substack{\uparrow \\ \text{old Hamiltonian}}}{\mathcal{H}'} = \frac{\partial S}{\partial t}$

$K=0$ is equivalent to HJ equation.

Note: we interpreted ~~P~~ E as some P. This is just an interpretation. Physical meaning of E is energy. For canonical transformation it is only important what is conjugated pair. And since $K=0$, both E and its conjugate will be constant ~~of~~ in time t.

Denote conjugate of E ("P") as τ ("Q").

$$P_1 = \frac{\partial S}{\partial X_1} = \sqrt{2E_1 - (\omega^2+B^2)X_1^2}$$

$$P_2 = \frac{\partial S}{\partial X_2} = \sqrt{2E_2 - (\omega^2+B^2)X_2^2}$$

$$\tau_1 = \frac{\partial S}{\partial E_1} = -t + \frac{1}{\sqrt{B^2+\omega^2}} \arcsin\left(\sqrt{\frac{B^2+\omega^2}{2E_1}}\, X_1\right)$$

$$\tau_2 = \frac{\partial S}{\partial E_2} = -t + \frac{1}{\sqrt{B^2+\omega^2}} \arcsin\left(\sqrt{\frac{B^2+\omega^2}{2E_2}}\, X_2\right)$$

So we get: $X_i = \sqrt{\frac{2E_i}{B^2+\omega^2}} \sin\left(\sqrt{B^2+\omega^2}\,(t+\tau_i)\right)$

and then it is easy to get $P_i = \sqrt{2E_i} \cos\left(\sqrt{B^2+\omega^2}\,(t+\tau_i)\right)$

Original coordinates movement is obtained by rotation:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos Bt & -\sin Bt \\ \sin Bt & \cos Bt \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2E_1}{B^2+\omega^2}} \sin\left(\sqrt{B^2+\omega^2}\,(t+\tau_1)\right) \\ \sqrt{\frac{2E_2}{B^2+\omega^2}} \sin\left(\sqrt{B^2+\omega^2}\,(t+\tau_2)\right) \end{pmatrix}$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = \begin{pmatrix} \cos Bt & -\sin Bt \\ \sin Bt & \cos Bt \end{pmatrix} \begin{pmatrix} \sqrt{2E_1}\ \cancel{\cos(\sqrt{B^2+\omega^2}}\ \cos\left(\sqrt{B^2+\omega^2}\,(t+\tau_1)\right) \\ \sqrt{2E_2} \cos\left(\sqrt{B^2+\omega^2}\,(t+\tau_2)\right) \end{pmatrix}$$

11. There are two movements in this motion

periods of

$$T_B = \frac{2\pi}{B} \qquad\qquad T_\Omega = \frac{2\pi}{\Omega}, \quad \Omega = \sqrt{B^2+\omega^2}$$

If $\frac{T_B}{T_\Omega}$ is a rational number, then after sufficiently long Δt system will return to its original position, i.e. trajectory will be closed

More precisely; if $\frac{T_B}{T_\Omega} = \frac{m}{n}$, then after $\Delta t = k \cdot T_B$, wher k is divisible by n, the system will return to original position.

You can download online example of a plot for a computer.