HW5: Canonical transformations and Hamilton-Jacobi equation

If you find misprints, have any questions, find some task difficult and want a hint, contact me by email vel145@gmail.com.

Canonical transformations

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1. Let $S_1(q,Q,t)$, $S_2(q,P,t)$, $S_3(p,Q,t)$, $S_4(p,P,t)$ be the four generating functions of the canonical transformation. Express

   (a) p and P, and $\mathcal{H}' - \mathcal{H}$ as the derivatives of $S_1(q,Q,t)$,

   (b) p and Q, and $\mathcal{H}' - \mathcal{H}$ as the derivatives of $S_2(q,P,t)$,

   (c) q and P, and $\mathcal{H}' - \mathcal{H}$ as the derivatives of $S_3(p,Q,t)$,

   (d) q and Q, and $\mathcal{H}' - \mathcal{H}$ as the derivatives of $S_4(p,P,t)$.

   Sign conventions follow from $dS_1 = pdq - PdQ - (\mathcal{H} - \mathcal{H}')dt$.

2. Find canonical transformations $Q = Q(q,p,t), P = P(q,p,t)$ generated by the functions below and explain their meaning. Here 3-dimensional vector $\mathbf{a}$ and parameters m, τ, t_1, t_2 are all some fixed numbers. Find answers only in a linear approximation with respect to $\mathbf{a}$ and τ.

   (a) $S_3(p,Q) = \sum_{i=1}^{3} \left(-p_i Q^i - a^i p_i\right)$,

   (b) $S_2(q,P) = \sum_{i,j,k=1}^{3} \left(q^i P_i - \epsilon_{ijk} a^i q^j P_k\right)$,

   (c) $S_3(p,Q,t) = -p\,Q - \tau\,\mathcal{H}(p,Q,t)$,

   (d) $S_1(q,Q) = \frac{m}{2}\frac{(Q-q)^2}{t_2 - t_1}$.

3. Find action $S(q_f, q_i, t)$ for a one-dimensional free particle of mass m as a function of its initial and final positions and time. Treating $q_f = q$ and $q_i = Q$, find the canonical transformation generated by this action.

4. For what values of α and β the following transformation is canonical

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\begin{aligned}
X &= \frac{1}{\alpha}\left(\sqrt{2p_x}\sin x + \beta\, p_y\right), & P_X &= +\frac{\alpha}{2}\left(\sqrt{2p_x}\cos x - \beta\, y\right), \\
Y &= \frac{1}{\alpha}\left(\sqrt{2p_x}\cos x + \beta\, y\right), & P_Y &= -\frac{\alpha}{2}\left(\sqrt{2p_x}\sin x - \beta\, p_y\right)?
\end{aligned}
\tag{1}
$$

   You may use any method you like. Typical approaches are to compute Poisson brackets $\{Q^i, Q^j\}, \{P_i, P_j\}, \{Q^i, P_j\}$ or to show that $p_i\,dq^i - P_i\,dQ^i$ is an exact differential.

Hamilton-Jacobi equation

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5. Check that $S(q,Q,t)$ from question 3 satisfies the Hamilton-Jacobi equation.

6. Solve the Hamilton-Jacobi equation for a one-dimensional free particle of mass m by the separation of variables method (as on the lecture, consider an ansatz $S(q,t) = W(q,E) - E\,t$). Take all integrals explicitly. Show that $W(q,E)$ is the Legendre transform of $S(q,Q,t)$ from question 3.

The following questions are devoted to solving the Hamilton-Jacobi equation for the 2-dimensional harmonic oscillator in a constant magnetic field B which is described by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + B\,(\dot{x}\,y - x\,\dot{y}) - \frac{1}{2}\omega^2(x^2 + y^2)\,. \tag{2}$$

7. Write the Lagrangian in the first-order form: $\mathcal{L} = p_x\,\dot{x} + p_y\,\dot{y} - \mathcal{H}$.

8. Perform the following time-dependent canonical transformation (a rotation by the angle $B\,t$)

$$\begin{aligned} x &= +X\,\cos B\,t - Y\,\sin B\,t\,, & p_x &= +P_X\,\cos B\,t - P_Y\,\sin B\,t\,,\\ y &= +X\,\sin B\,t + Y\,\cos B\,t\,, & p_y &= +P_X\,\sin B\,t + P_Y\,\cos B\,t\,. \end{aligned} \tag{3}$$

   Find the new Hamiltonian and first-order Lagrangian, show that they describe two decoupled harmonic oscillators.

   Note: note that Hamiltonian is not a scalar function under time-dependent transformation, i.e. it is not enough to simply substitute x, y, p_x, p_y with X, Y, P_X, P_Y in the expression for $\mathcal{H}$. Recall the answer from question 1

9. Write and solve the Hamilton-Jacobi equation for the new variables. You may use $\int dx\sqrt{a^2 - b^2\,x^2} = \frac{1}{2}x\,\sqrt{a^2 - b^2x^2} + \frac{a^2}{2b}\arcsin\left(\frac{b}{a}\,x\right)$.

10. Find the motion in the original coordinates.

11. [extra point]. Plot the trajectory of this motion on the computer in the $\{x, y\}$ plane. Choose several different values of B and initial conditions. For which values of B the trajectory is closed, i.e. particle returns to its original location after some time?