<u>Note</u>: I do things more detailed than you were requested, so you can learn by going through my solutions

<u>HW3</u>

<u>Solutions</u>

①

Phase portrait for

① $\dot{x} = x + y$
$\dot{y} = x - y$

- Canonical approach.

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix} \quad ; \quad A = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

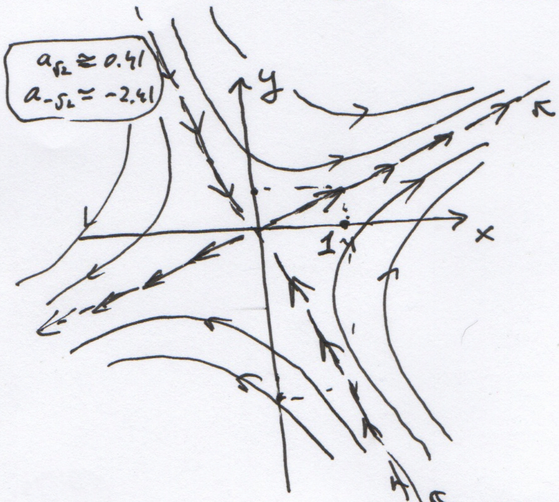
$$\left.\begin{aligned} \det A &= \lambda_1\lambda_2 = -2 \\ \operatorname{tr} A &= \lambda_1 + \lambda_2 = 0 \end{aligned}\right\} \Rightarrow \lambda_1 = -\lambda_2 = \sqrt{2} \quad \leftarrow \boxed{\text{saddle point}}$$

~~it is easy to guess~~

derive eigenvectors: $A\begin{pmatrix}1\\ a\end{pmatrix} = \pm\sqrt{2}\begin{pmatrix}1\\ a\end{pmatrix}$

1st row: $1 + a = \pm\sqrt{2}$; $a = \pm\sqrt{2} - 1$

check 2nd row $1 - a = 1 \mp \sqrt{2} + 1 = \mp\sqrt{2} = -a$ ✓



← asymptote with $\lambda = \sqrt{2}$ ↑ positive

← asymptote with $\lambda = -\sqrt{2}$ ↙ negative

"Hamiltonian" approach

- "~~Non canonical~~". ~~Saddle points satis~~ with $\operatorname{tr}A = 0$

Vector field near saddle points satisfies

$\operatorname{div}\vec{v} = 0$, because $\operatorname{tr}A = 0$.

Hence it satisfies Liouville theorem

<u>in 2D</u> it is enough for the field to be Hamiltonian

$$\left.\begin{aligned} \dot{x} &= \frac{\partial \mathcal{H}}{\partial y} \\ \dot{y} &= -\frac{\partial \mathcal{H}}{\partial x} \end{aligned}\right\} \Rightarrow \left.\frac{\partial \mathcal{H}}{\partial y}\right|_x = x + y \longrightarrow \mathcal{H} = \frac{1}{2}y^2 + yx + h(x)$$

arbitrary function of x

$$\left.\frac{\partial \mathcal{H}}{\partial x}\right|_y = y - x \qquad \frac{\partial \mathcal{H}}{\partial x} = y + \frac{dh}{dx}$$

~~$\mathcal{H} = \frac{1}{2}x^2 + yx - \frac{1}{2}y^2 = \frac{1}{2}(x+y)^2$~~

By comparing $\frac{\partial \mathcal{H}}{\partial x} = y - x$ and $\frac{\partial \mathcal{H}}{\partial x} = y + \frac{dh}{dx}$

1-cont-ed

we conclude $h = -\frac{1}{2}x^2$

Hence: $\boxed{\mathcal{H} = \frac{1}{2}y^2 + yx - \frac{1}{2}x^2}$

Hamiltonian flows preserve Hamiltonian.

H = const is a trajectory of the phase portrait

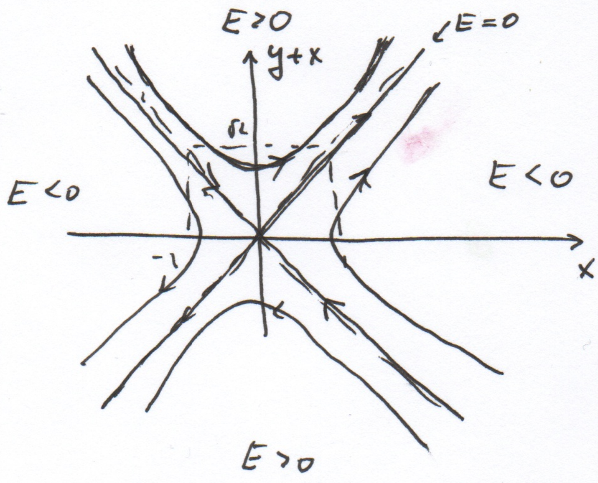
$$\frac{1}{2}y^2 + yx - \frac{1}{2}x^2 - E = 0$$

$$y^2 + 2yx - x^2 - 2E = 0$$

$$y = \frac{-2x + \sqrt{4x^2 + 4(x^2+2E)}}{2} = -x + \sqrt{2(x^2+E)}$$

$\boxed{y = -x \pm \sqrt{2(x^2+E)}}$ ← is explicit trajectory of phase portrait

$E = 0$: $y = -x \pm \sqrt{2}x = x(-1 \pm \sqrt{2})$ ← asymptotes



(a lot like relativistic picture)

(2) $\dot{x} = y$
$\dot{y} = 3x^2 - 1$

• "Canonical approach"

We should find zeros of the vector field $\begin{pmatrix} y \\ 3x^2-1 \end{pmatrix}$ and expand with linear approximation

$$\begin{matrix} y=0 \\ 3x^2-1=0 \end{matrix} \rightarrow \left\{ x = \pm \frac{1}{\sqrt{3}} \,;\, y = 0 \right\}$$

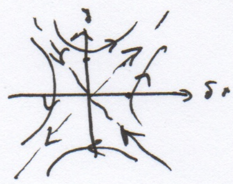
Expansion near point $\{x = +1/\sqrt{3}; y=0\}$: $x = 1/\sqrt{3} + \delta x$

$$\dot{\delta x} = y$$
$$\dot{y} = 3(x - 1/\sqrt{3})(x + 1/\sqrt{3}) \simeq 3\delta x \cdot \frac{2}{\sqrt{3}} = 2\sqrt{3} \cdot \delta x$$

Hence we have $A = \begin{pmatrix} 0 & 1 \\ 2\sqrt{3} & 0 \end{pmatrix}$ in $\begin{pmatrix} \delta\dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} \delta x \\ y \end{pmatrix}$

$$\left.\begin{matrix} \mathrm{tr} A = 0 \\ \det A = -2\sqrt{3} \end{matrix}\right\} \Rightarrow \underbrace{\lambda_1 = -\lambda_2}_{\text{saddle point}} = \sqrt{2\sqrt{3}}$$

eigen vectors $\begin{pmatrix} 1 \\ \pm\sqrt{2\sqrt{3}} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ \pm 1.86 \end{pmatrix}$

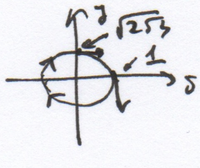


Expansion near point $\{x = -1/\sqrt{3}; y = 0\}$: $x = -1/\sqrt{3} + \delta x$

$$\delta\dot{x} = y$$
$$\delta\dot{y} = 3(x - 1/\sqrt{3})(x + 1/\sqrt{3}) = -2\sqrt{3}\,\delta x \;;\; A = \begin{pmatrix} 0 & 1 \\ 2\sqrt{3} & 0 \end{pmatrix} \qquad \underbrace{\lambda_1 = -\lambda_2 = i\sqrt{2\sqrt{3}}}_{\text{center}}$$

Note that

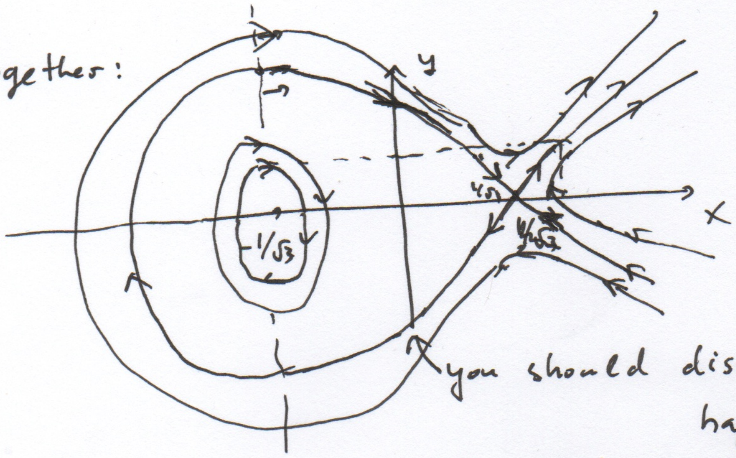
$\alpha(\delta x)^2 + y^2$ should be integral of motion



$$\frac{d}{dt}\left(\alpha(\delta x)^2 + y^2\right) = 2\alpha\,\delta x \cdot y + 2y \cdot (-2\sqrt{3})\,\delta x \rightarrow \alpha = 2\sqrt{3}$$

2-contred

Together:



you should discuss what happens with this line. [0.1 points]
It either closes on itself or goes to infinity.

At large negative x:

$\dot{x} = y$
$\dot{y} \cong 3x^2$, it shows that trajectories these are unbounded

- Hamiltonian approach.

$$\partial_x y + \partial_y (3x^2-1) = 0 \Rightarrow \text{this vector field is Hamiltonian}$$

$$\left.\frac{\partial \mathcal{H}}{\partial y}\right|_x = \dot{x} = y \quad \rightarrow \quad \mathcal{H} = \tfrac{1}{2} y^2 + h(x) \quad (*)$$

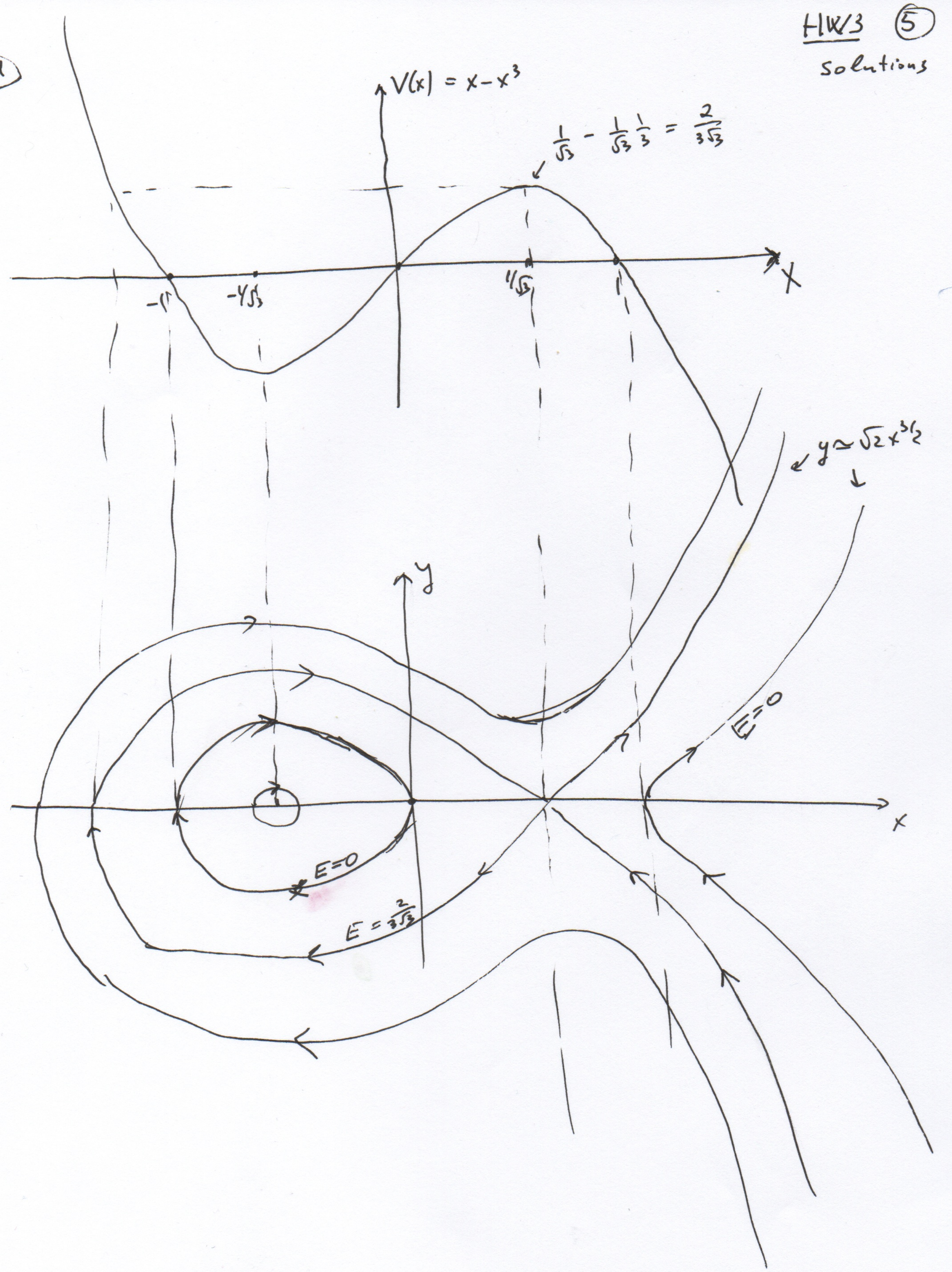
$$\left.\begin{array}{l} \left.\dfrac{\partial \mathcal{H}}{\partial x}\right|_y = -\dot{y} = 1 - 3x^2 \\ \text{from } (*):\ \left.\dfrac{\partial \mathcal{H}}{\partial x}\right|_y = \dfrac{dh}{dx} \end{array}\right\} \Rightarrow h = x - x^3$$

$$\boxed{\mathcal{H} = \frac{1}{2} y^2 + \underbrace{x - x^3}} \leftarrow \text{Hamiltonian of a particle in a potential!}$$

Trajectories are explicitly given by

$$y = \sqrt{2(E-V)} = \sqrt{2(E - x + x^3)}$$

2-cont-ed


$V(x) = x - x^3$
$\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\frac{1}{3} = \frac{2}{3\sqrt{3}}$
-1
$-1/\sqrt{3}$
$1/\sqrt{3}$
1
x
$y \simeq \sqrt{2}\, x^{3/2}$
y
$E=0$
$E=0$
$E = \frac{2}{3\sqrt{3}}$
x

(3) $\dot{x} = x^2 y - y$
$\dot{y} = -xy^2 + x$

• Canonical approach to find critical points.

$$\begin{cases} x^2 y - y = 0 \\ -xy^2 + x = 0 \end{cases} \Leftrightarrow \begin{cases} (x^2-1)y = 0 \\ (y^2-1)x = 0 \end{cases}$$

Solutions: $\begin{matrix} x=+1 \\ y=+1 \end{matrix}$; $\begin{matrix} x=+1 \\ y=-1 \end{matrix}$; $\begin{matrix} x=-1 \\ y=+1 \end{matrix}$; $\begin{matrix} x=-1 \\ y=-1 \end{matrix}$; $\begin{matrix} x=0 \\ y=0 \end{matrix}$

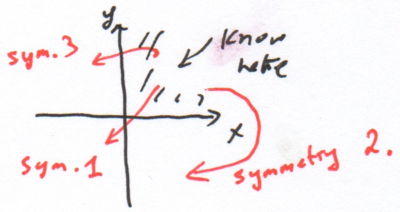
Analysing 5 points would take quite a time. [you can do this of course]
But in this case we can benefit from symmetry of equations:

Check that equations are invariant under:

1. $\begin{matrix} x \to -x \\ y \to -y \end{matrix}$ 2. $\begin{matrix} x \to -y \\ y \to x \end{matrix}$ 3. $\begin{matrix} x \to y \\ y \to -x \end{matrix}$

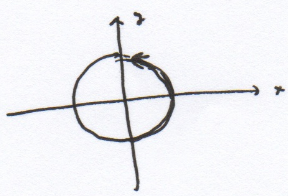
So, if we know solution in the 1st quadrant $\left(\begin{matrix} x>0 \\ y>0 \end{matrix}\right)$,

we know solution everywhere!



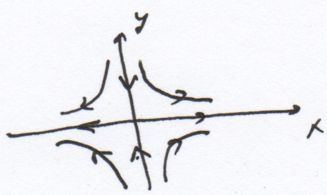
Hence, we need to analyze only $\begin{matrix} x=1 \\ y=1 \end{matrix}$ & $\begin{matrix} x=0 \\ y=0 \end{matrix}$

$\begin{matrix} x=0 \\ y=0 \end{matrix}$: $\begin{matrix} \dot{x} \simeq -y \\ \dot{y} \simeq +x \end{matrix}$ ← Harmonic oscillator!



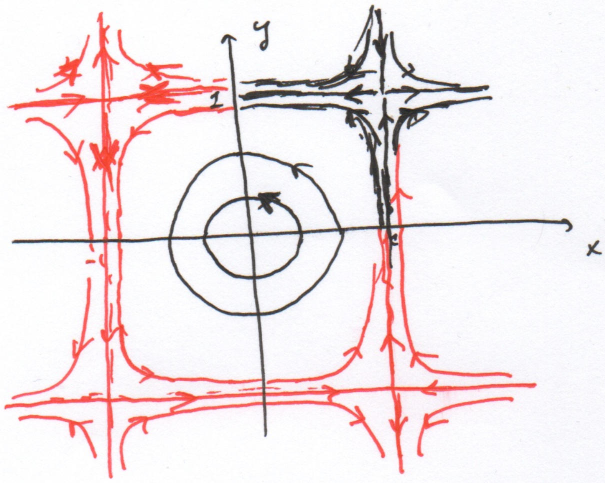
$\begin{matrix} x=1 \\ y=1 \end{matrix}$: $\left.\begin{matrix} \delta\dot{x} \simeq 2\delta x \\ \delta\dot{y} \simeq -2\delta y \end{matrix}\right\}$ ← saddle point

$x = 1 + \delta x$
$y = 1 + \delta y$



3 - continued

Together



it is important to check that $x = \pm 1$ ~~$y = \pm 1$~~ & $y = 1$ are straight lines exactly: [0.2 points]

$\dot{x} = y(x^2 - 1) = 0$ (for $x = 1$)
$\to x = 1$ (solution)

the same for $\dot{y}$

"Hamiltonian" approach

$$\operatorname{div} \vec{v} = \frac{\partial}{\partial x}(x^2 y - y) + \frac{\partial}{\partial y}(-xy^2 + x) = 2xy - 2xy = 0 \Rightarrow V \text{ is Hamiltonian v. field}$$

[Note: $\operatorname{div} \vec{v} = 0$ implies that $\vec{v}$ is Hamiltonian vector field only in 2d]

$$\left.\frac{\partial \mathcal{H}}{\partial y}\right|_x = (x^2 y - y) \Rightarrow \mathcal{H} = \frac{1}{2}x^2 y^2 - \frac{1}{2} y^2 + f(x)$$

$$\left.\frac{\partial \mathcal{H}}{\partial x}\right|_y = xy^2 - x \Rightarrow \mathcal{H} = \frac{1}{2}x^2 y^2 - \frac{1}{2}x^2 + \tilde{f}(y)$$

$$\Rightarrow \mathcal{H} = \frac{1}{2}(x^2 y^2 - y^2 - x^2)$$

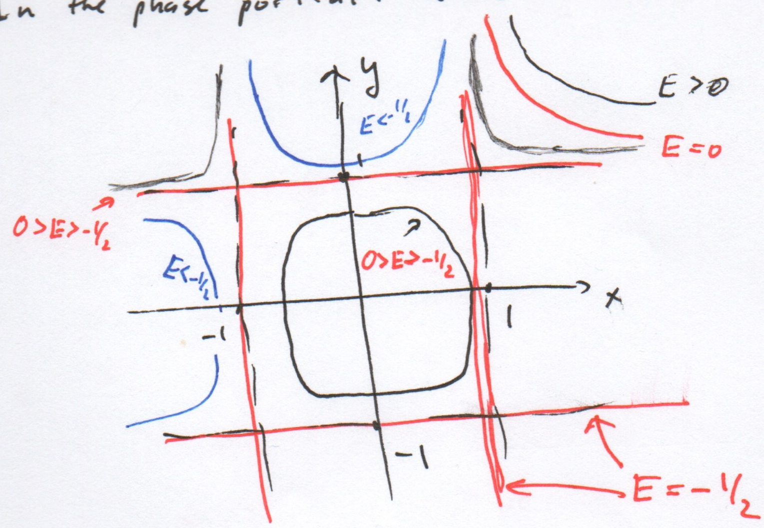
$\mathcal{H} = \text{const}$ are the trajectories of Hamiltonian flow.

$$E = \frac{1}{2}(x^2 y^2 + y^2 - x^2) \Rightarrow \boxed{y = \pm\sqrt{\frac{2E + x^2}{x^2 - 1}}}$$

explicit expression for trajectories.

In the phase portrait above:

HW3 solutions (8)



(4)

$$\dot{x} = y + x^2 + y^2 - 1$$
$$\dot{y} = -x + x^2 + y^2 - 1$$

; We note immediately that

$$\operatorname{div}\vec{v} = \frac{\partial}{\partial x}(y + x^2 + y^2 - 1) + \frac{\partial}{\partial y}(-x + x^2 + y^2 - 1) =$$

$$= 2x + 2y \neq 0 \Rightarrow$$ This vector field is not Hamiltonian

Critical points: $\begin{cases} y + x^2 + y^2 - 1 = 0 \\ -x + x^2 + y^2 - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} y = -x \\ 2x^2 = 1 + x \end{cases}$ ~~$\Leftrightarrow x = \pm 1/\sqrt{2},\ y = \mp 1/\sqrt{2}$~~

~~Expansion around: $x = 1/\sqrt{2} + \delta x$, $y = -1/\sqrt{2} + \delta y$; $\begin{pmatrix}\dot{\delta x} \\ \dot{\delta y}\end{pmatrix} = \hat{A}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix}$, $\hat{A} = \begin{pmatrix} & \end{pmatrix}$~~

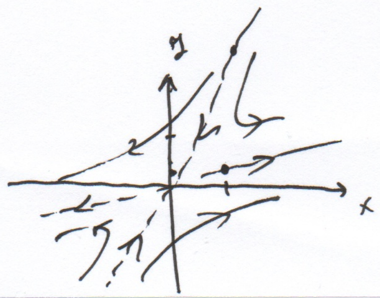
$$2x^2 - x - 1 = 0 \; ; \; x = \frac{1 \pm \sqrt{1+8}}{4} = \begin{cases} \frac{1+3}{4} = 1 \\ \frac{1-3}{4} = -1/2 \end{cases} \Rightarrow \begin{matrix} y = -1 \\ y = 1/2 \end{matrix}$$

<u>Expansion around</u> $x=1;\ y=-1$: $\quad x = 1 + \delta x$, $y = -1 + \delta y$

$$\begin{pmatrix}\delta\dot{x} \\ \delta\dot{y}\end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}}_{= \hat{A}}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix}$$

$$\left.\begin{matrix}\det A = -4 + 1 = -3 \\ \operatorname{tr} A = 0\end{matrix}\right\} \Rightarrow$$

$\Rightarrow \lambda_1 = -\lambda_2 = \sqrt{3}$ (saddle point)



eigenvectors: $\begin{pmatrix}1 \\ a\end{pmatrix} =$

$(2 - a) = \pm\sqrt{3}$

$a = 2 \mp \sqrt{3}$

check: $(1 - 2a) = 1 - 4 \pm 2\sqrt{3} = \pm\sqrt{3}(\mp\sqrt{3} + 2)$ ✓

4 - continued

Expansion around
$x = -\frac{1}{2}$
$y = \frac{1}{2}$:
$x = -\frac{1}{2} + \delta x$
$y = \frac{1}{2} + \delta y$

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}}_{= A} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

$$\left.\begin{matrix} \mathrm{tr}\, A = 0 \\ \det A = -1 + 4 = 3 \end{matrix}\right\} \Rightarrow \underbrace{\lambda_1 = -\lambda_2 = i\sqrt{3}}_{\text{center.}}$$

For centers we expect $C = (\delta x)^2 + A(\delta x\, \delta y) + B(\delta y)^2$ to be an integral of motion

$$\frac{dC}{dt} = 2\delta x\, \delta\dot{x} + A\delta\dot{x}\, \delta y + A\delta x\, \delta\dot{y} + 2B\, \delta y\, \delta\dot{y} =$$

$$= 2\delta x(-\delta x + 2\delta y) + A(-\delta x + 2\delta y)\delta y + A\delta x(-2\delta x + \delta y) +$$

$$+ B(-2\delta x + \delta y)\delta y =$$

$$= (\delta x)^2 \underbrace{(-2 - 2A)}_{\Downarrow\; A = -1} + (\delta y)^2 \underbrace{(2A + B)}_{\Downarrow\; B = -2A = 2} + \delta x\, \delta y \underbrace{(4 - A + A - 2B)}_{\Downarrow\; B = 2 \quad \text{ok!}}$$

$C = (\delta x)^2 - (\delta x\, \delta y) + 2(\delta y)^2$ is an integral of motion (for small fluctuations.

↑ this is an ellips.

Lagrange multiplier method:

let us find its furthest and closthest distance to origin and positions on ellips by method of Lagrange multiplier

$r^2 = (\delta x)^2 + (\delta y)^2$ ← want to find extrema for C = fixed

$$0 = \frac{\partial}{\partial \delta x}(r^2 + \lambda C) = 2\delta x + \lambda(2\delta x - \delta y) = \delta x(2 + 2\lambda) - \lambda \delta y$$

$$0 = \frac{\partial}{\partial \delta y}(r^2 + \lambda C) = 2\delta y + \lambda(-\delta x + 4\delta y) = -\lambda\delta x + \delta y(2 + 4\lambda)$$

$$\frac{\delta y}{\delta x} = \frac{2 + 2\lambda}{\lambda} \;;\; \frac{\delta y}{\delta x} = \frac{\lambda}{2 + 4\lambda} \;;\; \lambda^2 = (2 + 2\lambda)(2 + 4\lambda)$$

↓ Solutions

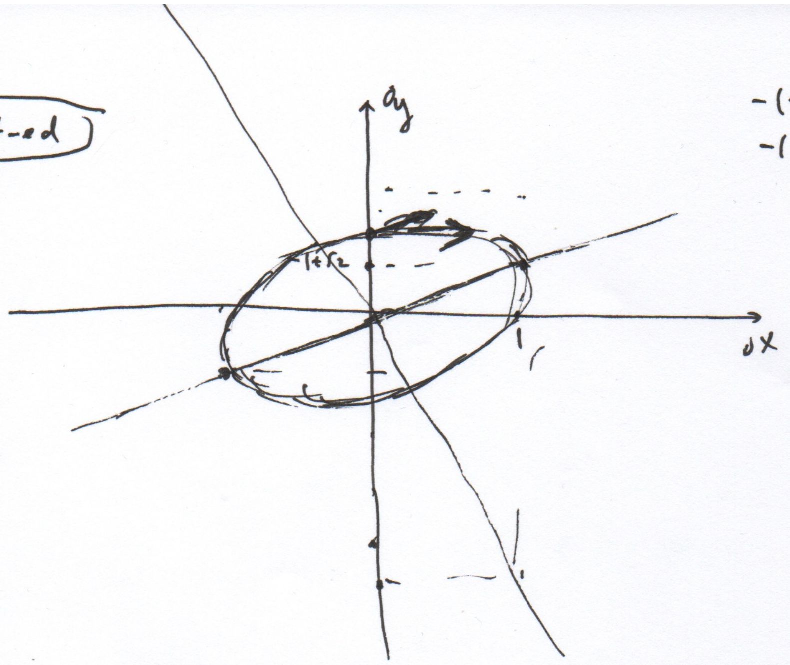
$$\frac{\delta y}{\delta x} = -1 \pm \sqrt{2} \quad \Leftarrow \quad \lambda = \frac{2}{7}(-3 \pm \sqrt{2})$$

(4-cont-ed)

HW 10 solutions

$-1+\sqrt{2} = 0.41$

$-1-\sqrt{2} = -2.41$

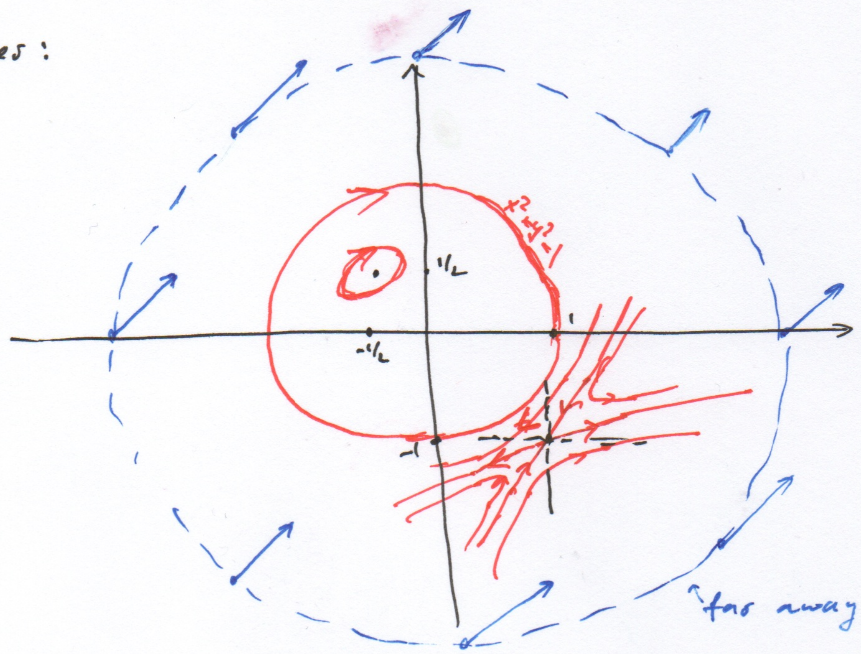


$$r^2 = \delta x^2 + \delta y^2 =$$
$$= \delta x^2 \left(1 + (1\mp\sqrt{2})^2\right) =$$
$$= \delta x^2 \left(2 \mp 2\sqrt{2} + 2\right) =$$
$$= \delta x^2 \left(4 \mp 2\sqrt{2}\right) = \delta x^2 \pm 2\sqrt{2}\left(\pm\sqrt{2}-1\right)$$

$$C = \delta x^2 - \delta x\,\delta y + 2\delta y^2 = (\delta x)^2\left(1 - \frac{\delta y}{\delta x} + 2\left(\frac{\delta y}{\delta x}\right)^2\right) =$$
$$= (\delta x)^2\left(1 - (-1\pm\sqrt{2}) + 2(-1\pm\sqrt{2})^2\right) =$$
$$= (\delta x)^2\left(2 \mp \sqrt{2} + 2 \mp 4\sqrt{2} + 4\right) = (\delta x)^2\left(8 \mp 5\sqrt{2}\right)$$

$$(\delta x)^2 = \frac{C}{8\mp 5\sqrt{2}}\,; \qquad r^2 = C\,\frac{4\mp 2\sqrt{2}}{8\mp 5\sqrt{2}} = \frac{2C}{7}\left(3\pm\sqrt{2}\right)$$

Together:



At $x, y \gg 1$:
$\dot{x} \cong \dot{y} \cong r^2$
[0.2 points]

Note interesting fact: $x^2+y^2=1$ is a trajectory!

Indeed
$$\dot{x} = y + \overbrace{x^2+y^2-1}^{0}$$
$$\dot{y} = -x + \underbrace{x^2+y^2-1}_{0}$$
has x^2+y^2 as integral of motion

4)- advanced discussion (can skip)

We learned that these are two critical points:

- $x=1$, $y=-1$ (saddle point)

  $x=-1/2$, $y=1/2$ (center)

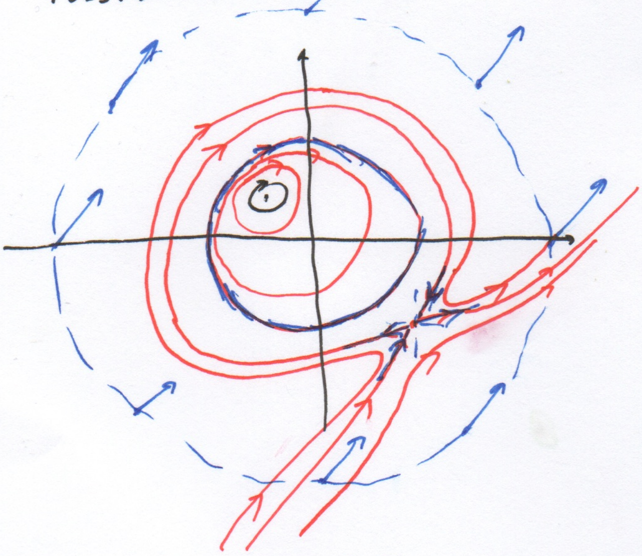
- and there is at least one big cycle $x^2+y^2=1$

- and also vector flow at infinity.

~~It is howe~~

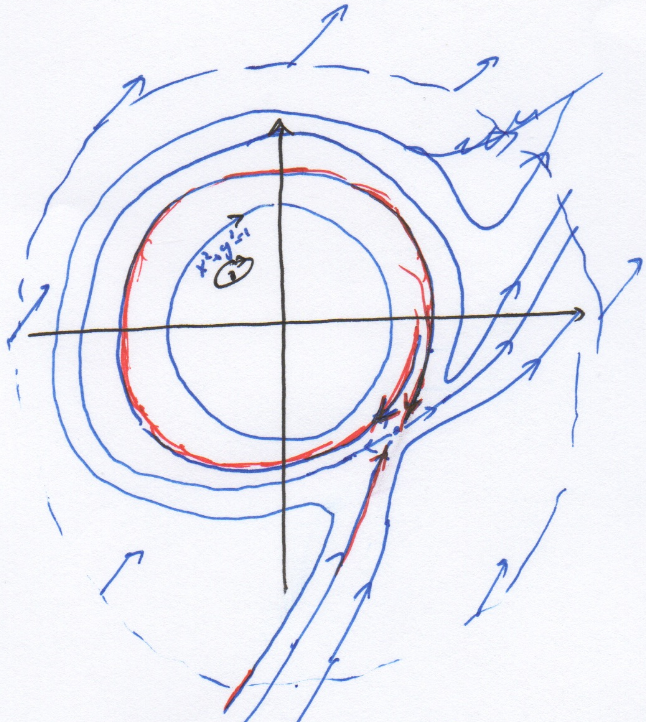
It is hower non-clear how to join theise knowledge into one picture

Possible variants:



[no spirals]

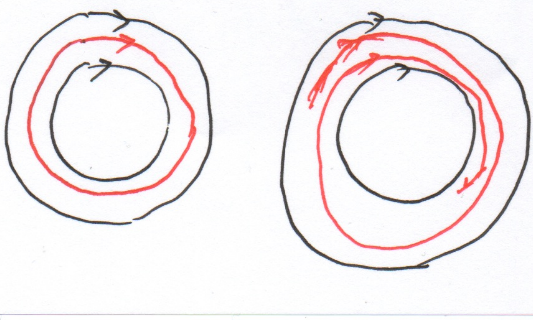
A



[there are spirals]

B

In general, if you have two closed cycles, in between you may have or not spirals:



[see Van der Pol oscillator] on wikipedia

4 - advanced discussion cont-ed

So, do we have spirals or not?

Actually, very remarkably, we can solve these equations exactly ???:

$$\dot{x} = y + x^2 + y^2 - 1$$
$$\dot{y} = -x + x^2 + y^2 - 1$$

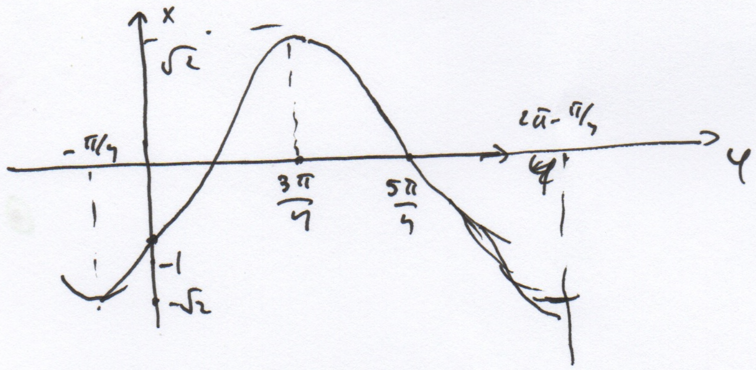
Go to ~~spherical~~ polar coordinates:

$$\dot{r} = (\cos\varphi + \sin\varphi)(r^2 - 1)$$
$$\dot{\varphi} = (\cos\varphi - \sin\varphi)\frac{r^2-1}{r} - 1$$

Since we only interested in trajectories, let us write equation for $\frac{dr}{d\varphi}$:

$$\frac{dr}{d\varphi} = \frac{(\cos\varphi + \sin\varphi)(r^2-1)}{(\cos\varphi - \sin\varphi)\frac{r^2-1}{r} - 1}$$

Introduce $x = \sin\varphi - \cos\varphi$
$dx = (\cos\varphi + \sin\varphi)\,d\varphi$



$$\frac{dr}{dx} = -\frac{r^2-1}{x\frac{r^2-1}{r} + 1}$$

Or: $\frac{dx}{dr} + \frac{x}{r} = -\frac{1}{r^2-1}$; Now note $\frac{dx}{dr} + \frac{x}{r} = \frac{1}{r}\frac{d}{dr}(xr)$

$\frac{d}{dr}(xr) = -\frac{r}{r^2-1} \rightarrow$ standart way of solving produces result:

$$\sin\varphi - \cos\varphi = x = -\frac{1}{2r}\log\left|\frac{1-r^2}{1-r_0^2}\right|$$; r_0 - constant of integration.

From here it is almost clear that $r(\varphi + 2\pi) = r(\varphi) \Rightarrow$ no spirals case $\boxed{A}$ is correct.

(5) 1., 2., 3. – Hamiltonian, Hamiltonians ~~were~~ were computed in 1., 2., 3.

1: $\mathcal{H} = \frac{1}{2} y^2 + yx - \frac{1}{2} x^2$

2: $\mathcal{H} = \frac{1}{2} y^2 + x - x^3$ ← particle in potential $V = x - x^3$

3: $\mathcal{H} = \frac{1}{2}(x^2y^2 - y^2 - x^2) = \frac{1}{2}\left(x^2(y^2-1) - (y^2-1) - 1\right) = \frac{1}{2}(x^2-1)(y^2-1) + \text{const}$

4. is not Hamiltonian, however trajectories can be found explictly:

~~Satisfy:~~

~~sinφ-cosφ~~

$$\boxed{x - y = \frac{1}{2} \log\left|\frac{1-(x^2+y^2)}{1-r_0^2}\right|}$$

r_0 is parameterises different trajectories.

---

comp ship: it is quite interesting, we can write

$$e^{2(x-y)} = C\left(1-(x^2+y^2)\right)$$

$$C = \frac{e^{2(x-y)}}{1-(x^2+y^2)}$$

. If we think about C as a Hamiltonian, it produces equations which have the same ~~solution~~ trajectories

$$\dot{x} = \frac{\partial C}{\partial y} = \frac{-2e^{2(x-y)}}{1-(x^2+y^2)} + \frac{2y e^{\omega}}{(1-(x^2+y^2))^2} = (y + x^2+y^2-1)\cdot 2\frac{e^{2(x-y)}}{1-(x^2+y^2)}$$

$$\dot{y} = -\frac{\partial C}{\partial x} = -\left(\frac{2e^{\omega}}{1-(x^2+y^2)} + \frac{2x e^{\omega}}{(1-(x^2+y^2))^2}\right) = (-x + x^2+y^2-1)\cdot 2\frac{e^{2(x-y)}}{1-(x^2+y^2)}$$

Puzzle for you: these equations are Hamiltonian, they preserve volume.

original equations ~~are not~~ do not preserve volume.

How this is possible?

6

We need to solve equation: [below $y \equiv p$]

HW3 14 solution

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0 \quad (1)$$

Because the vector flow is Hamiltonian, ~~this equation~~ $\operatorname{div}\vec{v} = 0$, the equation (1) can be rewritten as

$$\frac{\partial \rho}{\partial t} + \vec{v}\,\vec{\nabla}\rho = 0$$

$$\frac{\partial \rho}{\partial t} + V_x \frac{\partial \rho}{\partial x} + V_y \frac{\partial \rho}{\partial y} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \{\rho, \mathcal{H}\} = 0}$$

(recall that $V_x = \frac{\partial \mathcal{H}}{\partial y}$, $V_y = -\frac{\partial \mathcal{H}}{\partial x}$)

this is a useful but not absolutely necessary simplification.

a.) $\mathcal{H} = \frac{1}{2}(p^2 + x^2)$ $\boxed{\text{solution: } \rho = x^2 + p^2}$

we are proposed to use ansatz

$$\rho = a(t)(p^2 + x^2) + b(t)\,px$$

$$\{\rho, \mathcal{H}\} = b(t)\{px, \mathcal{H}\} = \frac{b(t)}{2}\{px, p^2 + x^2\} = b(t)(p^2 - x^2)$$

$$\frac{\partial \rho}{\partial t} = \dot{a}(p^2 + x^2) + \dot{b}\cdot px$$

$$\frac{\partial \rho}{\partial t} + \{\rho, \mathcal{H}\} = \dot{a}(p^2 + x^2) + \dot{b}\cdot px + b(p^2 - x^2) = 0$$

Since the last eqn should be equal to 0 for any p, x, It is compulsory that

$$\dot{a} = \dot{b} = b = 0 \Rightarrow \begin{matrix} b = 0 \\ a = \text{const} \end{matrix}$$

Hence: $\rho = a(p^2 + x^2)$

Since $\rho(t=0, x, p) = p^2 + x^2 \Rightarrow a = 1$

6 - continuation

therefore $\boxed{\rho(x,p,t) = p^2+x^2}$

We could derive it from the beginning. We know that $p^2+x^2 = 2\mathcal{H}$, and $\frac{d\mathcal{H}}{dt}=0 \Rightarrow \rho = 2\mathcal{H}$

b) $\mathcal{H} = \frac{1}{2}(p^2-x^2)$; $\boxed{\text{solution: } \rho = \cosh(2t)(p^2+x^2) - 2\sinh(2t)\,px}$

$$\rho = a(t)(p^2+x^2) + b(t)\,px$$

$$\{\rho, \mathcal{H}\} = \tfrac{1}{2}a(t)\{p^2+x^2, p^2-x^2\} + \tfrac{1}{2}b(t)\{px, p^2-x^2\} =$$

$$= \cancel{\tfrac{1}{2}}\,a(t)\{x^2,p^2\} + \tfrac{1}{2}b(t)\left(2p^2+2x^2\right) =$$

$$= 4a(t)xp + b(t)(p^2+x^2)$$

$$\frac{\partial\rho}{\partial t} = \dot{a}(x^2+p^2) + \dot{b}\cdot px$$

$$\frac{\partial\rho}{\partial t} + \{\rho,\mathcal{H}\} = (\dot{a}+b)(p^2+x^2) + (\dot{b}+4a)xp = 0$$

Since should be true for $\forall p, x$, one has:

$$\begin{cases}\dot{a} = -b\\ \dot{b} = -4a\end{cases} ; \quad \begin{pmatrix}\dot{a}\\ \dot{b}\end{pmatrix} = \begin{pmatrix}0 & -1\\ -4 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}$$

$$\left.\begin{array}{l}\operatorname{tr}A = 0\\ \det A = -4\end{array}\right\} \Rightarrow \lambda_1 = -\lambda_2 = -2$$

$$a = C_1 e^{2t} + C_2 e^{-2t}$$
$$b = D_1 e^{2t} + D_2 e^{-2t}$$

$$\dot{a} = 2C_1 e^{2t} - 2C_2 e^{-2t} = -D_1 e^{2t} - D_2 e^{-2t}$$

$$2C_1 = -D_1 ; \quad +2C_2 = D_2$$

$$b = -2C_1 e^{2t} + 2C_2 e^{-2t}$$

<u>check</u>: $\dot{b} = -4C_1 e^{2t} - 4C_2 e^{-2t} = -4a$ ✓

Hence solution is:

$$\rho = \left(C_1 e^{2t} + C_2 e^{-2t}\right)(p^2+x^2) + \left(-2C_1 e^{2t} + 2C_2 e^{-2t}\right)px$$

$$\rho(x,p,0) = p^2 + x^2 \Rightarrow \begin{cases} -2C_1 e^{2\cdot 0} + 2C_2 e^{-2\cdot 0} = 0 \\ C_1 e^{2\cdot 0} + C_2 e^{-2\cdot 0} = 1 \end{cases}$$

$$\begin{cases} -2C_1 + 2C_2 = 0 \\ C_1 + C_2 = 1 \end{cases} \Rightarrow C_1 = C_2 = 1/2$$

$$\boxed{\rho = \cosh(2t)(p^2+x^2) - \cancel{Ke^{2t}}\, 2\sinh(2t)\, p x}$$

Now we check that indeed $\frac{d\rho}{dt} = 0$

Equation of motion: $\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = x$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = p$$

$$\begin{aligned} \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \dot{x}\frac{\partial \rho}{\partial x} + \dot{p}\frac{\partial \rho}{\partial p} = {} & 2\sinh(2t)(p^2+x^2) - 4\cosh(2t)\, p x \\ & + p\cdot\left(2\cosh(2t)\cdot x - 2\sinh(2t)\cdot p\right) \\ & + x\left(2\cosh(2t)\cdot p - 2\sinh(2t)\cdot x\right) = 0 \end{aligned}$$

✓