HW3: Vector flows

If you find misprints, have any questions, find some task difficult and want a hint, contact me by email vel145@gmail.com.

Phase portraits

You may find useful to look on this vector field plotter: http://www.uwec.edu/smithaj/vfa/. Phase portraits can be drawn (almost nicely) in *Mathematica*, with the command `StreamPlot` (see help there for usage). Note, however, you should to not only draw the pictures but to demonstrate your capacity to draw them from scratch (i.e. to show how you look for critical points, their type etc).

Draw the phase portraits of the following vector flows

1. [1 point] $\begin{aligned} \dot{x} &= x + y\,, \\ \dot{y} &= x - y\,. \end{aligned}$
2. [1 point] $\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= 3x^2 - 1\,. \end{aligned}$
3. [2 points] $\begin{aligned} \dot{x} &= +x^2\, y - y\,, \\ \dot{y} &= -x\, y^2 + x\,. \end{aligned}$
4. [2 points] $\begin{aligned} \dot{x} &= +y + x^2 + y^2 - 1\,, \\ \dot{y} &= -x + x^2 + y^2 - 1\,. \end{aligned}$

Hamiltonian flows and Liouville's theorem

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5. [2 points] Which of the vector flows from questions 1-4 can be generated by a Hamiltonian? In the cases it can, find the Hamiltonian.

   Note: Symplectic structure is assumed to be the canonical one. I.e. identify $x \equiv q, y \equiv p$.

6. [2 points] Let $\rho(p, x, 0) = p^2 + x^2$ be a density distribution at the initial moment of time. By solving the continuity equation, find $\rho(p, x, t)$ at any moment of time, if the vector flow is generated by a) Hamiltonian $\mathcal{H} = \frac{1}{2}(p^2 + x^2)$, b) Hamiltonian $\mathcal{H} = \frac{1}{2}(p^2 - x^2)$. Check explicitly that the obtained density $\rho(x, y, t)$ is an integral of motion.

   Hint: Try an ansatz $\rho(p, x, t) = a(t)(p^2 + x^2) + b(t)\, p\, x$.