HW2: Exterior algebra and differential forms

Recommended reading is Aronld's book, chapter 7 "Differential forms". Please do not hesitate to contact me, we can discuss its contents.

If you find misprints, have any questions, find some task difficult and want a hint, contact me by email vel145@gmail.com.

Exterior algebra

The exterior algebra is also discussed on Wikipedia.

Any n-dimensional vector space V (take $\mathbb{R}^n$ for simplicity) can be thought of as the space of homogeneous linear functions of n variables. Indeed, if these variables are $\mathbf{x}_i$, any such function can be represented as

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots a_n\mathbf{x}_n\,. \tag{1}$$

One can think about $\mathbf{x}$'s just as the basis vectors!

Consider now the tensor product $V\otimes V$. It has dimension n^2, and any its element is of the form $\sum a_{ij}\mathbf{x}_i\otimes\mathbf{x}_j$.

What if we don't want to distinguish the order of terms in $\mathbf{x}_i\otimes\mathbf{x}_j$, i.e we want to identify $\mathbf{x}_i\otimes\mathbf{x}_j = \mathbf{x}_j\otimes\mathbf{x}_i$? It is possible; a space with such identification is called symmetric square of V, and is denoted by $S^2(V)$.

What if we want to identify $\mathbf{x}_i\otimes\mathbf{x}_j = -\mathbf{x}_j\otimes\mathbf{x}_i$? It is possible; a space with such identification is called exterior square of V, and is denoted by $\Lambda^2(V)$.

1. Prove that $S^2(V)$ and $\Lambda^2(V)$ are vector spaces.

$S^2(V)$ is isomorphic to the space of degree 2 homogeneous polynomials in n variables. Indeed, identification $\mathbf{x}_i\otimes\mathbf{x}_j = \mathbf{x}_j\otimes\mathbf{x}_i$ is equivalent to saying that we multiply two commuting variables: $\mathbf{x}_i\,\mathbf{x}_j$.

What about $\Lambda^2(V)$? To describe it, we can introduce new-type variables θ^i which anti-commute. They are called Grassmann variables or Grassmann numbers though they are not numbers in the usual sense. They satisfy the property:

$$\theta^i\wedge\theta^j = -\theta^j\wedge\theta^i\,. \tag{2}$$

The symbol $\wedge$ denotes the product in the algebra of Grassmann variables and is called wedge product.

A simple consequence of (2) is

$$\theta^i\wedge\theta^i \equiv \left(\theta^i\right)^2 = 0\,. \tag{3}$$

By continuing the logic, one can consider fully symmetric tensors from $V^{\otimes k}$, this space is known as symmetric power $S^k(V)$, and the fully antisymmetric tensors from $V^{\otimes k}$, this space is known as exterior power $\Lambda^k(V)$. While the former space is isomorphic to the space of degree k homogeneous polynomials in commuting variables, the latter one is isomorphic to the space of degree k homogeneous polynomials in Grassmann variables.

All possible polynomials of n Grassmann variables (not necessarily homogeneous, including constant terms for instance) is called **Grassmann** or **exterior** algebra. It is obviously an algebra because it is a vector space and multiplication is well-defined by (2). It is denoted by $\Lambda(V)$.

2. Get used to this algebra by expanding the following expressions:

(a) $(5\theta_1 + 3\theta_2) \wedge (\theta_1 - 2\theta_3 + 1)$

(b) $(\theta_1 - \theta_2)^2$

(c) $(\theta_1 + \theta_2 + \theta_3)^2$

(d) $(\theta_1 + \theta_2 + \theta_3 + \theta_4)^5$

(e) $e^{\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4}$ (consider exponent to be defined by its Taylor series)

3. Prove that this algebra is finite-dimensional. What is dimension of $\Lambda^k(V)$? What is dimension of $\Lambda(V)$?

Below $\alpha, \beta \in \Lambda^1(V)$, $\omega, \eta \in \Lambda^2(V)$. For instance, $\alpha = \alpha_i \theta^i$, $\omega = \sum_{i<j} \omega_{ij} \theta^i \wedge \theta^j = \frac{1}{2}\omega_{ij}\theta^i \wedge \theta^j$.

4. Let $X = \alpha \wedge \beta$, $Y = \alpha \wedge \omega$, $Z = \omega \wedge \eta$. Write down explicitly the components X_{ij}, Y_{ijk} and Z_{ijkl}.

5. Show that if $\alpha \wedge \omega = 0$ then $\omega = \alpha \wedge \beta$ for some β.

6. Is it true that if $\omega \wedge \omega = 0$ then $\omega = \alpha \wedge \beta$ for some α and β? If yes, prove. If not, give a counter-example. Let ω_{ij} be coordinates in $n(n-1)/2$-dimensional space. Define a subspace in it by $\omega \wedge \omega = 0$. What is the dimension of this subspace?

7. Let n be even. Write down explicitly $\omega^{n/2}$ (in components). How is answer related to $\det\limits_{1\leq i,j\leq n} \omega_{ij}$? If in doubt, try $n = 2$ and $n = 4$ first.

Differential forms

If V from discussion above is actually a V^*, a linear functional on some vector space, then elements of $\Lambda^k(V^*)$ are called exterior forms of degree k, or **k-forms**. The k-form is naturally a linear functional on the elements of $V^{\otimes k}$. **Differential k-form** at each point x of the manifold M is an exterior k-form, with $V^* = T^*M_x$ being the dual space to tangent space at point x. In the case of differential form one uses notation dx^i instead of θ^i.

Differential forms are explicitly parameterised as follows

$$\begin{aligned}
\omega_{(0)} &= f(x)\,,\text{0-form is just a function (scalar field)}\,,\\
\omega_{(1)} &= \alpha_i(x)\,dx^i\,,\\
\omega_{(2)} &= \sum_{i<j}\omega_{ij}(x)dx^i\wedge dx^j = \frac{1}{2}\omega_{ij}(x)dx^i\wedge dx^j\,,\\
&\dots\\
\omega_{(k)} &= \sum_{i_1<i_2\ldots<i_k}\omega_{i_1 i_2\ldots i_k}(x)dx^{i_1}\wedge\ldots dx^{i_k} = \frac{1}{k!}\omega_{i_1 i_2\ldots i_k}(x)dx^{i_1}\wedge\ldots dx^{i_k}\,. \qquad (4)
\end{aligned}$$

8. Consider the case of $n = 3$. Consider three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Find the value of $dx^1 \wedge dx^2 \wedge dx^3$ on $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$. Prove that this value is equal to the euclidean volume of parallelogram defined by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

9. How the components of the k-form transform under the change of coordinates $\mathbf{y} = \mathbf{y}(\mathbf{x})$? Write the answer in analogy with the case of 1-forms discussed on the lecture. Can you write the answer in compact form for the case of n-forms (reminder: n is dimension of the vector space, as everywhere in this space)?

10. Rewrite $dx \wedge dy \wedge dz$ in spherical coordinates and $dx \wedge dy$ in complex coordinates.

11. Let g be the determinant of the metric. For definition of the metric see e.g. question 9 of Tutorial 1. Prove that $dV = \sqrt{g} dx^1 \wedge dx^2 \wedge \ldots dx^n$ is invariant under coordinate transformations. This is the so-called volume form.

Differentiation and integration

De Rahm differential d is an operator defined as

$$d \equiv dx^i \wedge \frac{\partial}{\partial x^i} \,. \tag{5}$$

The convention is that partial derivatives do not act on dx's, only on components $\omega_{i_1 \ldots}(x)$ in (4). This is more clear when we use θ's instead of dx's.

Explicitly, for a 1-form α:

$$d\alpha = dx^i \wedge (\partial_i \alpha_j dx^j) = \frac{1}{2} \left(\partial_i \alpha_j - \partial_j \alpha_i\right) dx^i \wedge dx^j \,, \tag{6}$$

where we anti-symmetrised $\partial_i \alpha_j$ in the second equality using that $dx^i \wedge dx^j = -dx^j \wedge dx^i$.

12. Compute $d\left(z^2\, dx \wedge dy + e^x dx \wedge dz + (x-y)^3\, dy \wedge dz\right)$. Be careful about signs.

13. Write down in components, like in (6), $d\omega$, where ω is a 2-form.

14. Prove that

    (a) d is a linear operator

    (b) $d(\omega_{(k)} \wedge \omega_{(l)}) = (d\omega_{(k)}) \wedge \omega_{(l)} + (-1)^k \omega_{(k)} \wedge (d\omega_{(l)})$

    (c) $d\, d = 0$

15. Fromulate definition of integral of the differential k-form. Show how the generic Stokes formula[1]

$$\int_{\partial \mathrm{D}} \omega = \int_{\mathrm{D}} d\omega \tag{7}$$

reduces to: a) Green's formula b) Gauss-Ostrogradsky theorem. Basically you need to comprehend the contents of the Wikipedia page "Stokes' theorem".

[1]derivation of the Stokes formula is not required.