# MA2342 - Homewrk 1 - Solutions and comments

The Answer/Solution parts of this text are essentially based on the solution submitted in TeX by Jean Lagacé. I would like to give special thanks to him for providing his homework in the LaTeX format.

# 1 Differential forms

#### Problem 1

QUESTION: Integrate  $\alpha = (x^3 - 2x^2y + xy^2 - 2y^3) dx - 2x dy$  over the unit circle, in a counterclockwise direction. Explain the obtained result.

ANSWER :

$$\oint_{\mathbb{S}^1} (x^3 - 2x^2y + xy^2 - 2y^3) \, \mathrm{d}x - 2x \, \mathrm{d}y = 0$$

SOLUTION:

Let us use the following parametrisation for the circle for  $t \in [-\pi, \pi]$ :

$$\begin{aligned} x &= \cos t, & y &= \sin t; \\ dx &= -\sin t \, dt, & dy &= \cos t \, dt; \end{aligned}$$

which yields the following integral:

$$\int_{-\pi}^{\pi} -\cos^3 t \sin t + 2\cos^2 t \sin^2 t - \cos t \sin^3 t + 2\sin^4 t - 2\cos^2 t \,\mathrm{d}t.$$
(1.1)

The first and the third term integrate to 0 since they are odd functions. Using the identities  $\sin^2 t + \cos^2 t = 1$  and using the fact that the integral of  $\sin^2 t - \cos^2 t$  over the whole period is 0, by symmetry, we get that this integrates to 0.

#### DISCUSSION:

- You can use NIntegrate command in *Mathematica* to numerically compute (1.1) and to check your answer in this way. This example is very simple of course and computer can do it analytically as well, but you will face more challenging cases in your research. You should acquire a habit of checking your results, in particular numerically.
- One could think that  $\oint \alpha = 0$  implies that  $\alpha$  is an exact differential. However, we cannot make any statement on the exactness of  $\alpha$  based on this computation, it simply means that integrating on this specific curve is 0. Sparing the computation, if we had integrated the same differential on the square  $\partial [0, 1]^2$ , the result would have been -1/6. It is also easy to see, from the usual criterion on the exactness of differentials,  $\frac{\partial \alpha_x}{\partial y} = \frac{\partial \alpha_y}{\partial x}$ , that it cannot be equal to df for some f, and is therefore not exact.

• However, on the contour of integration condition  $x^2 + y^2 = 1$  holds. It allows us to simplify the integrand to (2x - 2y) dx - 2x dy. The latter one is exact and non-singular inside the contour of integration. Hence the integral should be zero.

#### Problem 2

QUESTION: If you have a force **F** acting on a particle, you can define the differential form  $F_i dx^i$  (summation over *i* is assumed). Then the work over a path  $\gamma$  is given by  $\int_{\gamma} F_i dx^i$ . Translate the notion of "conservative" and "non-conservative" force to the language of differential forms.

ANSWER: Conservative force is by definition a force whose work does not depend on a path but only on the starting and ending points. This is one of the necessary and sufficient conditions for a differential to be exact. So a conservative force is an exact differential, while a non-conservative force is a non-exact differential.

DISCUSSION: Another name for conservative force is potential force: F = -dV, where V is potential (minus sign is convention). Note that usually we consider F as a vector, not as a differential form. In this case we take a gradient,  $\mathbf{F} = -\nabla V$  or explicitly  $F^i = -g^{ij}\frac{\partial V}{\partial x^j}$ , as discussed on the lecture. So the third name for the same thing is gradient force.

# 2 Hamiltonians

Let us first state the equation for the Hamiltonian and the Hamiltonian equations of motion. The Hamiltonian is the Legendre transform of the Lagrangian with respect to the  $\dot{q}$  variables, that is it is given by

$$\mathcal{H} = \dot{q}^i p_i - \mathcal{L} \,,$$

where  $p_i$  are the generalised momenta given by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$

From there, we can get the Hamiltonian equations of motion, a set of 2n first order differential equations, where n is the number of degrees of freedom, as

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}$$

# Problem 3

QUESTION: Find the Hamiltonian and Hamilton's equations of motion for a system with the following Lagrangian

$$\mathcal{L} = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + B(x_1\dot{x}_2 - x_2\dot{x}_1) - U(x_1, x_2).$$
(2.1)

Note: In the original task the potential was written with the opposite sign. It is ok if you did your computations coherently, all up to the end with the opposite sign.

ANSWER: The Hamiltonian is

$$\mathcal{H} = \frac{1}{m} \left[ \frac{1}{2} \left( p_1^2 + p_2^2 \right) - B \left( x_1 p_2 - x_2 p_1 \right) + \frac{B^2}{2} \left( x_1^2 + x_2^2 \right) \right] + U(x_1, x_2).$$

The equations of motion are

$$\dot{p}_1 = \frac{B \, p_2 - B^2 \, x_1}{m} - \frac{\partial U}{\partial x_1}; \qquad \dot{p}_2 = \frac{-B \, p_1 - B^2 \, x_2}{m} - \frac{\partial U}{\partial x_2}; \\ \dot{x}_1 = \frac{1}{m} \left( p_1 + B x_2 \right); \qquad \dot{x}_2 = \frac{1}{m} \left( p_2 - B x_1 \right).$$

SOLUTION: First, let us compute the momenta,

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m \dot{x}_1 - B x_2;$$
  
$$p_2 = \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m \dot{x}_2 + B x_1;$$

then we can get an expression for both  $\dot{x}_1$  and  $\dot{x}_2$ ,

$$\dot{x}_1 = \frac{1}{m} (p_1 + Bx_2);$$
  
 $\dot{x}_2 = \frac{1}{m} (p_2 - Bx_1).$ 

All is left to do is to write the Hamiltonian, following the equation at the beginning of the section:

$$\mathcal{H} = \frac{1}{m} \left[ p_1^2 + Bp_1 x_2 + p_2^2 - Bp_2 x_1 - \frac{1}{2} \left( p_1^2 + 2Bp_1 x_2 + B^2 x_2^2 + p_2^2 - 2Bp_2 x_1 + B^2 x_1^2 \right) - Bx_1 p_2 + B^2 x_1^2 + Bx_2 p_1 + B^2 x_2^2 \right] + U(x_1, x_2).$$

Simplifying this mess, we get

$$\mathcal{H} = \frac{1}{m} \left[ \frac{1}{2} \left( p_1^2 + p_2^2 \right) - B \left( x_1 p_2 - x_2 p_1 \right) + \frac{B^2}{2} \left( x_1^2 + x_2^2 \right) \right] + U(x_1, x_2).$$

Finally, we get the equations of motion, including those for  $\dot{x}$  we already computed,

$$\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = \frac{Bp_2 - B^2 x_1}{m} - \frac{\partial U}{\partial x_1}; \qquad \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} - \frac{Bp_1 + B^2 x_2}{m} - \frac{\partial U}{\partial x_2};$$
$$\dot{x}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = \frac{1}{m} \left( p_1 + Bx_2 \right); \qquad \dot{x}_2 = \frac{\partial \mathcal{H}}{\partial p_2} = \frac{1}{m} \left( p_2 - Bx_1 \right).$$

## Problem 4

QUESTION: Rewrite the Lagrangian from the previous exercise in polar coordinates and find the Hamiltonian and Hamilton's equations of motion in terms of generalised coordinates and momenta:  $\{r, \phi, p_r, p_{\phi}\}$ .

ANSWER : The Lagrangian is

$$\mathcal{L} = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) + Br^2 \dot{\theta} - U(r, \theta).$$

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \left[ p_r^2 + \frac{1}{r^2} (p_\theta - B r^2)^2 \right] + U.$$

The equations of motion are

$$\dot{p}_r = \frac{1}{m} \left( \frac{p_\theta^2}{r^3} - r B^2 \right) - \frac{\partial U}{\partial r}; \qquad \dot{p}_2 = -\frac{\partial U}{\partial \theta};$$
  
$$\dot{r} = \frac{p_r}{m}; \qquad \dot{\theta} = \frac{p_\theta - Br^2}{mr^2}.$$

SOLUTION: To change into polar coordinates, we use the following change of variables :

$$\begin{aligned} x_1 &= r \cos \theta, & x_2 &= r \sin \theta; \\ \dot{x}_1 &= \dot{r} \cos \theta - \dot{\theta} r \sin \theta, & \dot{x}_2 &= \dot{r} \sin \theta + \dot{\theta} r \cos \theta; \end{aligned}$$

which transforms the Lagrangian into

$$\mathcal{L} = \frac{m}{2} \left( \dot{r}^2 \cos^2 \theta - 2\dot{r}\dot{\theta} \cos \theta \sin \theta + \dot{\theta}^2 r^2 \sin^2 \theta + \dot{r}^2 \sin^2 \theta + 2\dot{r}\dot{\theta} \cos \theta \sin \theta + \dot{\theta}^2 r^2 \cos^2 \theta \right) + B \left( \dot{r}r \sin \theta \cos \theta + \dot{\theta}r^2 \cos^2 \theta - \dot{r}r \sin \theta \cos \theta + \dot{\theta}r^2 \sin^2 \theta \right) + U(r,\theta);$$

and becomes, after simplification

$$\mathcal{L} = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) + Br^2 \dot{\theta} - U(r, \theta).$$

We can now compute the Hamiltonian. To do so, we must first compute the momenta, yielding

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r};$$
  
$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} + Br^2.$$

From this, we can get expressions for  $\dot{r}$  and  $\dot{\theta}$  as

$$\dot{r} = \frac{p_r}{m};$$
  
$$\dot{\theta} = \frac{1}{mr^2} \left( p_\theta - Br^2 \right);$$

and we write the Hamiltonian

$$\mathcal{H} = p_r \dot{r} + p_\theta \dot{\theta} - \mathcal{L} = p_r \frac{p_r}{m} + p_\theta \frac{1}{mr^2} \left( p_\theta - Br^2 \right) - \frac{m}{2} \left( \frac{p_r^2}{m^2} + \frac{1}{m^2 r^2} \left( p_\theta - Br^2 \right)^2 \right) - Br^2 \frac{1}{mr^2} \left( p_\theta - Br^2 \right) + U$$

which gets simplified to

$$\mathcal{H} = \frac{1}{2m} \left[ p_r^2 + \frac{1}{r^2} (p_\theta - B r^2)^2 \right] + U.$$

We finally get the equations of motion as follow, using what we already computed for  $\dot{r}$  and  $\dot{\theta}$ 

$$\dot{p}_r = \frac{1}{m} \left( \frac{p_{\theta}^2}{r^3} - r B^2 \right) - \frac{\partial U}{\partial r}; \qquad \dot{p}_2 = -\frac{\partial U}{\partial \theta};$$
$$\dot{r} = \frac{p_r}{m}; \qquad \dot{\theta} = \frac{p_{\theta} - Br^2}{mr^2}.$$

One can observe from these equations of motion that if the potential is radial, it means that  $p_{\theta}$  is a conserved quantity for a system with this Hamiltonian.

DISCUSSION: It is good and important that you do an explicit computation similar to above. At the same time, learn to observe several generic patterns. There were several of them during this computation:

• It is quite often that kinetic term is quadratic in velocities:  $\frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j$ . The matrix g should be sought as a metric. Passing from one coordinate frame (original  $\{x, y\}$  coordinates) to another (polar coordinates) means changing metric. If you start from a quadratic quantity like  $\frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j$ , you will end up with a quadratic quantity  $\frac{1}{2}\tilde{g}_{ij}\dot{q}^i\dot{q}^j$ , where by  $\tilde{q}$  and  $\tilde{g}$  we denoted the corresponding objects in the new coordinate frame. In this particular case we can find  $\tilde{g}$  (almost) without computation. First, we note that polar coordinates is an orthogonal frame hence g should be diagonal (think about this! can you explain these statements?). Hence the kinetic term can be at most of the form  $\frac{1}{2}\left(A(r,\phi)\dot{r}^2 + B(r,\phi)\dot{\phi}^2\right)$ . Due to rotational symmetry of the problem, A and B should be independent from  $\phi$ . Purely by dimensional analysis, A = const,  $B = r^2 \text{const}$ . It remains to fix two numerical constants. Find by yourself two arguments that demonstrate that these constants are both equal to 1 (consider velocity vectors in some special directions when the answer is under very good control).

• If you make Legendre transform of a quadratic form, you end up with a quadratic form defined by an inverse matrix:

$$\frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j \to (\text{Legendre transform}) \to \frac{1}{2}(g^{-1})^{ij}p_i\,p_j\,. \tag{2.2}$$

Note that it is completely for g depend on co-ordinates. It should not depend however on the velocities (i.e. objects which are subject to the Legendre transform).

• If your Lagrangian has a term linear in velocities:

$$\mathcal{L} = F(q)\dot{q} + \mathcal{L}' \tag{2.3}$$

Then you can do the following: 1) Find the Hamiltonian for  $\mathcal{L}'$  2) Replace in this Hamiltonian  $p \to p - F(q)$ , and you will get the right answer (Prove this fact! Or understand the mechanism behind on the example of Problem 4). Note: this technique is used in the next problem.

I must say I don't encourage you to use these properties as the ONLY mean for deriving your answer. Formulated in generic fashion, they might render your computation too formalised so you would not have a grip on it. The best is to do the computation partially boldly, and then to check whether your answer can be obtained from the abovementioned generic observations as well. Then you have two independent ways to obtain the result and you get a firm confidence of its correctness if both ways give the same answer.

#### Problem 5

QUESTION: Find the Hamiltonian and Hamilton's equations of motion for a relativistic particle of charge e which is moving in the constant magnetic field **B**. The lagrangian is given by:

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{2c} \epsilon_{\alpha\beta\gamma} B_{\alpha} x_{\beta} v_{\gamma} , \qquad (2.4)$$

where  $v_{\alpha} = \dot{x}_{\alpha}$  is the velocity vector and v is its absolute value.  $\epsilon_{\alpha\beta\gamma}$  is the fully antisymmetric tensor with  $\epsilon_{123} = 1$ .

ANSWER: The Hamiltonian is

$$\mathcal{H} = mc^2 \sqrt{1 + \frac{P^2}{m^2 c^2}} \,,$$

where  $P_{\alpha} = p_{\alpha} - \frac{e}{2c} \epsilon_{\alpha\beta\gamma} B_{\beta} x_{\gamma}$ . The equations of motion are

$$\begin{split} \dot{x}_{\alpha} &= \frac{1}{\sqrt{1 + \frac{P^2}{m^2 c^2}}} \frac{P_{\alpha}}{m} \,, \\ \dot{p}_{\alpha} &= -\frac{1}{\sqrt{1 + \frac{P^2}{m^2 c^2}}} \frac{\epsilon_{\alpha\beta\gamma} B_{\beta} P_{\gamma}}{m} \end{split}$$

SOLUTION: We get  $p_{\alpha}$  by differentiating with respect to  $v_i$ , that is

$$p_{\alpha} = \frac{\partial \mathcal{L}}{\partial v^{\alpha}} = \frac{mv_{\alpha}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{2c} \epsilon_{\alpha\beta\gamma} B_{\beta} x_{\gamma} \,.$$

We used the property  $\epsilon_{\alpha\beta\gamma} = \epsilon_{\gamma\alpha\beta}$ . Note also that  $A_{\alpha} = \epsilon_{\alpha\beta\gamma}B_{\beta}C_{\gamma}$  means in the vector notation  $\mathbf{A} = \mathbf{B} \times \mathbf{C}$  (check this!). So actually we can write in the index free form:

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{2c}\mathbf{B} \times \mathbf{r}$$
(2.5)

In what follows we introduce notation  $\mathbf{P} \equiv \mathbf{p} - \frac{e}{2c}\mathbf{B} \times \mathbf{r}$ . One should keep in mind that  $\mathbf{P}$  depends on r. But for the sake of Legendre transform this is the only quantity that matters.

Solving for  $\mathbf{v}$  one has

$$\mathbf{v} = \frac{\mathbf{P}}{m} \sqrt{1 - \frac{v^2}{c^2}}.$$

We can find  $v^2$  from computing the norm of the last expression ( $v^2 \equiv \mathbf{v} \cdot \mathbf{v}$  and  $P^2 \equiv \mathbf{P} \cdot \mathbf{P}$ ):

$$v^2 = \frac{P^2}{m^2} \left( 1 - \frac{v^2}{c^2} \right),$$

which after some algebra gives us

$$v^2 = \frac{1}{1 + \frac{P^2}{m^2 c^2}} \frac{P^2}{m^2}, \quad 1 - \frac{v^2}{c^2} = \frac{1}{1 + \frac{P^2}{m^2 c^2}}.$$

From there, we can get  ${\bf v}$  and the Lagrangian as

$$\begin{split} \mathbf{v} &= \frac{\mathbf{P}}{m} \frac{1}{\sqrt{1 + \frac{P^2}{m^2 c^2}}} \,, \\ \mathcal{L} &= \frac{-mc^2}{\sqrt{1 + \frac{P^2}{m^2 c^2}}} + \frac{e}{2c} \mathbf{v} \cdot \left( \mathbf{B} \times \mathbf{r} \right), \end{split}$$

and we can get the Hamiltonian as

$$\mathcal{H} = \mathbf{p} \cdot \mathbf{v} - \mathcal{L} = (\mathbf{P} + \frac{e}{2c} \mathbf{B} \times \mathbf{r}) \cdot \mathbf{v} - \mathcal{L} = \mathbf{P} \cdot \mathbf{v} + \frac{mc^2}{\sqrt{1 + \frac{P^2}{m^2c^2}}}$$
$$= \frac{1}{\sqrt{1 + \frac{P^2}{m^2c^2}}} \left(\frac{P^2}{m} + mc^2\right) = mc^2 \sqrt{1 + \frac{P^2}{m^2c^2}}.$$
(2.6)

To get the equations of motion, let us first compute the derivative of  $P^2$  with respect to  $x_{\alpha}$ .

$$\frac{\partial P^2}{\partial x_{\alpha}} = -2P_{\gamma} \left(\frac{e}{2c}\epsilon_{\gamma\beta\alpha}B_{\beta}\right) = 2\frac{e}{2c}\epsilon_{\alpha\beta\gamma}B_{\beta}P_{\gamma},$$

where we used  $\epsilon_{\alpha\beta\gamma} = -\epsilon_{\gamma\beta\alpha}$ . In index-free notation the last expression reads:

$$\nabla P^2 = 2\frac{e}{2c}\mathbf{B} \times \mathbf{P} \,. \tag{2.7}$$

From this, we recover the  $\dot{x}_i$  that we already computed as

$$\dot{x}_{\alpha} = \frac{\partial \mathcal{H}}{\partial p_{\alpha}} = \frac{P_{\alpha}}{m} \frac{1}{\sqrt{1 + \frac{P^2}{m^2 c^2}}}$$

and we compute  $\dot{p}$  as

$$\dot{\mathbf{p}} = -\nabla \mathcal{H} = -\frac{e}{2mc} \frac{1}{\sqrt{1 + \frac{P^2}{m^2 c^2}}} \mathbf{B} \times \mathbf{P}.$$

DISCUSSION: If you are not familiar enough with Levi-Civita symbol, you might find it useful to work in vector index-free notations. Another option is to choose explicitly  $\mathbf{B}$ along z-direction and write down all the formulae explicitly.

There are two important checks to do about your result. First, when  $\mathbf{B} = 0$ , one has  $\mathbf{P} = \mathbf{p}$  and the Hamiltonian becomes the standard relativistic expression for the energy of moving particle, equations of motion are also in place. Second, consider  $c \to \infty$  and put  $B_x = B_y = 0$ . Then we find ourselves essentially in the set up of problem 3 with  $\frac{e}{2c}B_z = B$ . For the latter we know explicitly equations of motion, so we can check our findings in problem 5<sup>1</sup>.

# 3 Poisson Brackets

We define the Poisson bracket, using Einstein summation convention, as

$$\{f,g\} = \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$$

where p and q are Darboux coordinates.

# Problem 6

QUESTION: Prove that  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$ 

<sup>&</sup>lt;sup>1</sup>When preparing this solution, I did this limit to check myself and found a mistake in my text! Do this limit as well, it is always worth to do such checks on your computation

SOLUTION #1: Applying the definition of the Poisson bracket, we get that

$$\{f, \{g,h\}\} = \frac{\partial f}{\partial q_j} \left[ \frac{\partial^2 g}{\partial q_i \partial p_j} \frac{\partial h}{\partial p_i} + \frac{\partial g}{\partial q_i} \frac{\partial^2 h}{\partial p_i \partial p_j} - \frac{\partial^2 g}{\partial p_i \partial p_j} \frac{\partial h}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial^2 h}{\partial q_i \partial p_j} \right]$$
$$- \frac{\partial f}{\partial p_j} \left[ \frac{\partial^2 g}{\partial q_i \partial q_j} \frac{\partial h}{\partial p_i} + \frac{\partial g}{\partial q_i} \frac{\partial^2 h}{\partial p_i \partial q_j} - \frac{\partial^2 g}{\partial p_i \partial q_j} \frac{\partial h}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial^2 h}{\partial q_i \partial q_j} \right];$$

where we assumed that f,g,h are differentiable enough that the second derivative commutes. We can then write

$$\{f, \{g, h\}\} = \frac{\partial f}{\partial q_j} \frac{\partial^2 g}{\partial q_i \partial p_j} \frac{\partial h}{\partial p_i} + \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial q_i} \frac{\partial^2 h}{\partial p_i \partial p_j} + \frac{\partial f}{\partial p_j} \frac{\partial^2 g}{\partial q_j \partial p_i} \frac{\partial h}{\partial q_i} + \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial p_i} \frac{\partial^2 h}{\partial q_i \partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial^2 g}{\partial p_i \partial p_j} \frac{\partial h}{\partial q_i} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_i} \frac{\partial^2 h}{\partial q_i \partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial^2 g}{\partial q_j \partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_i} \frac{\partial^2 h}{\partial p_i \partial q_j}.$$

Similarly, we can compute the two other ones as

$$\{g, \{h, f\}\} = \frac{\partial g}{\partial q_j} \frac{\partial^2 h}{\partial q_i \partial p_j} \frac{\partial f}{\partial p_i} + \frac{\partial g}{\partial q_j} \frac{\partial h}{\partial q_i} \frac{\partial^2 f}{\partial p_i \partial p_j} + \frac{\partial g}{\partial p_j} \frac{\partial^2 h}{\partial q_j \partial p_i} \frac{\partial f}{\partial q_i} + \frac{\partial g}{\partial p_j} \frac{\partial h}{\partial p_i} \frac{\partial^2 f}{\partial q_i \partial q_j} - \frac{\partial g}{\partial q_j} \frac{\partial^2 h}{\partial p_i \partial p_j} \frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_j} \frac{\partial h}{\partial p_i} \frac{\partial^2 f}{\partial q_i \partial p_j} - \frac{\partial g}{\partial p_j} \frac{\partial^2 h}{\partial q_j \partial q_i} \frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_j} \frac{\partial h}{\partial q_i} \frac{\partial^2 f}{\partial p_i \partial q_j};$$

and

$$\{h, \{f,g\}\} = \frac{\partial h}{\partial q_j} \frac{\partial^2 f}{\partial q_i \partial p_j} \frac{\partial g}{\partial p_i} + \frac{\partial h}{\partial q_j} \frac{\partial f}{\partial q_i} \frac{\partial^2 g}{\partial p_i \partial p_j} + \frac{\partial h}{\partial p_j} \frac{\partial^2 f}{\partial q_j \partial p_i} \frac{\partial g}{\partial q_i} + \frac{\partial h}{\partial p_j} \frac{\partial f}{\partial p_i} \frac{\partial^2 g}{\partial q_i \partial q_j} - \frac{\partial h}{\partial q_j} \frac{\partial^2 f}{\partial p_i \partial p_j} \frac{\partial g}{\partial q_i} - \frac{\partial h}{\partial q_j} \frac{\partial f}{\partial p_i} \frac{\partial^2 g}{\partial q_i \partial p_j} - \frac{\partial h}{\partial p_j} \frac{\partial^2 f}{\partial q_j \partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial h}{\partial p_j} \frac{\partial f}{\partial q_i} \frac{\partial^2 g}{\partial p_i \partial q_j}.$$

It then becomes a simple matter of summing those three expressions, and seeing that every term appears summed once and subtracted once. The fact that we could switch the derivatives at will using Fubini's theorem had to be use, the identity is not true otherwise.

SOLUTION #2: As you can see above, each of the explicitly written terms has one factor which is a second derivative. Landau-Lifshitz book has a simple argument why all second derivatives should cancel out. Since there are no terms free of them, the whole expression should be zero.

SOLUTION #3: Define the differential operator  $D_f \equiv \frac{\partial f}{\partial q} \frac{\partial}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial}{\partial q}$ . First, we need to convince ourself that  $\{f, g\} = [D_f, D_g] \cdot 1$ , where  $[A, B] \equiv AB - BA$  is what is standardly known as commutator and  $\cdot 1$  means that one acts on the identity function. The next step would be rather straightforward:

$$\{f, \{g, h\}\} = [D_f, [D_g, D_h]] \cdot 1.$$
(3.1)

Then the Jacobi identity follows from the well-known Jacobi identity for commutators:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \qquad (3.2)$$

the latter is straightforwardly check by expanding commutators using their definition  $[A, B] \equiv AB - BA$ .

## Problem 7

QUESTION: The Hamiltonian of a free particle in D dimensions is given by  $\mathcal{H} = \sum_{i=1}^{D} \frac{p_i^2}{2m}$ . Consider the quantity  $J_{ij} = p_i x_j - p_j x_i$ . Compute the following poisson brackets:

$$\{J_{ij}, p_k\}, \{J_{ij}, x_k\}, \{J_{ij}, J_{kl}\},$$
(3.3)

in the case of  $\{J_{ij}, J_{kl}\}$ , express the answer as a linear combination of J's only. What is the value of  $\{J_{14}, J_{13}\}$ ?

Prove that  $J_{ij}$  is the conserved quantity by computing  $\{J_{ij}, \mathcal{H}\}$ .

 $\{J_{ij}, p_k\}$ 

ANSWER:

$$\{J_{ij}, p_k\} = \delta_{jk}p_i - \delta_{ik}p_j.$$

First write out the expression as

$$\{J_{ij}, p_k\} = \frac{\partial J_{ij}}{\partial x_\alpha} \frac{\partial p_k}{\partial p_\alpha} - \frac{\partial J_{ij}}{\partial p_\alpha} \frac{\partial p_k}{\partial x_\alpha}$$

and note that any  $p_k$  is independent of all  $q_\alpha$  and of  $p_\alpha$  if  $\alpha \neq k$ . Therefore,

$$\frac{\partial p_k}{\partial x_\alpha} = 0 \,\forall \alpha, \qquad \frac{\partial p_k}{\partial p_\alpha} = \delta_{k\alpha};$$

where  $\delta_{ij}$  is the Kronecker delta. Therefore, we can rewrite our first equation as

$$\{J_{ij}, p_k\} = \delta_{k\alpha} \frac{\partial J_{ij}}{\partial x_{\alpha}},$$
$$= \frac{\partial J_{ij}}{\partial x_k}.$$

From the expression for  $J_{ij}$ , we finally get that

$$\{J_{ij}, p_k\} = \delta_{jk} p_i - \delta_{ik} p_j.$$

# $\{J_{ij}, x_k\}$ ANSWER:

$$\{J_{ij}, x_k\} = \delta_{jk} x_i - \delta_{ik} x_j$$

Similarly for this one, we have that

$$\frac{\partial x_k}{\partial p_\alpha} = 0 \,\forall \alpha, \qquad \frac{\partial x_k}{\partial x_\alpha} = \delta_{k\alpha};.$$

This yields that

$$\{J_{ij}, x_k\} = \frac{\partial J_{ij}}{\partial x_\alpha} \frac{\partial x_k}{\partial p_\alpha} - \frac{\partial J_{ij}}{\partial p_\alpha} \frac{\partial x_k}{\partial x_\alpha}$$
$$= -\delta_{k\alpha} \frac{\partial J_{ij}}{\partial p_\alpha}$$
$$= \delta_{jk} x_i - \delta_{ik} x_j$$

 $\{J_{ij}, J_{kl}\}$ ANSWER:

$$\{J_{ij}, J_{kl}\} = \delta_{jk}J_{il} + \delta_{il}J_{jk} - \delta_{jl}J_{ik} - \delta_{ik}J_{jl}.$$

$$\{J_{13}, J_{14}\} = -J_{34}$$

Let us first expand the expression for  $J_{ij}$  and  $J_{kl}$  in the Poisson bracket as

$$\{J_{ij}, J_{kl}\} = \frac{\partial(p_i x_j - p_j x_i)}{\partial x_{\alpha}} \frac{\partial(p_k x_l - p_l x_k)}{\partial p_{\alpha}} - \frac{\partial(p_i x_j - p_j x_i)}{\partial p_{\alpha}} \frac{\partial(p_k x_l - p_l x_k)}{\partial x_{\alpha}}$$

Using once again the respective independance of the  $x_a$ s and of the  $p_a$ s, we gets

$$\begin{aligned} \{J_{ij}, J_{kl}\} &= (\delta_{j\alpha}p_i - \delta_{i\alpha}p_j)(\delta_{k\alpha}x_l - \delta_{l\alpha}x_k) - (\delta_{j\alpha}x_i - \delta_{i\alpha}x_j)(\delta_{k\alpha}p_l - \delta_{l\alpha}p_k) \\ &= \delta_{jk}p_ix_l - \delta_{jl}p_ix_k - \delta_{ik}p_jx_l + \delta_{il}p_jx_k - \delta_{jk}p_lx_i + \delta_{jl}p_kx_i + \delta_{ik}p_lx_i - \delta_{il}p_kx_i, \end{aligned}$$

where we used the fact that  $\delta_{ij}\delta_{ik} = \delta_{jk}$ . Rearranging terms, this leaves us with

$$\{J_{ij}, J_{kl}\} = \delta_{jk}J_{il} + \delta_{il}J_{jk} - \delta_{jl}J_{ik} - \delta_{ik}J_{jl}.$$

This, in turn allows us to easily compute  $\{J_{13}, J_{14}\}$  as

$$\{J_{13}, J_{14}\} = -J_{34}$$

 $\{J_{ij}, \mathcal{H}\}$ ANSWER :

$$\{J_{ij},\mathcal{H}\}=0$$

and therefore it is a conserved quantity.

We know that a quantity is conserved in the Hamiltonian equations of motion when the Poisson bracket of that quantity with the Hamiltonian is equal to 0. Let us compute the Poisson bracket of  $J_{ij}$  with the Hamiltonian in the case of the free particle.

$$\{J_{ij},\mathcal{H}\} = \frac{\partial(p_i x_j - p_j x_i)}{\partial x_\alpha} \frac{\partial \mathcal{H}}{\partial p_\alpha} - \frac{\partial(p_i x_j - p_j x_i)}{\partial p_\alpha} \frac{\partial \mathcal{H}}{\partial x_\alpha}.$$

Using the fact that the free particle Hamiltonian is independent of x, we further get

$$\{J_{ij}, \mathcal{H}\} = (\delta_{j\alpha}p_i - \delta_{i\alpha}p_j) \frac{\partial \mathcal{H}}{\partial p_{\alpha}};$$
$$= \frac{p_i p_j}{m} - \frac{p_j p_i}{m};$$
$$= 0;$$

which ends up proving that  $J_{ij}$  is a conserved quantity in the free particle problem.

# Problem 8

QUESTION: For the case of free particle in D=3, express angular momenta  $M_i$  in terms of  $J_{jk}$  using  $\epsilon_{ijk}$ . And vice versa, express J's in terms of M's. Find the Poisson brackets

$$\{M_i, p_k\}, \{M_i, x_k\}, \{M_i, M_k\}.$$
 (3.4)

If possible, express the answer in terms of M again. ANSWERS:

 $M_i$  in terms of  $J_{jk}$  and vice versa are given by

$$M_{i} = -\frac{1}{2}\epsilon_{ijk}J_{jk}$$
$$J_{jk} = -\epsilon_{jki}M_{i} = \epsilon_{kji}M_{i}$$

The various Poisson brackets we had to compute were

$$\{M_i, x_j\} = \epsilon_{ijk} x_k$$
  
$$\{M_i, p_j\} = \epsilon_{ijk} p_k$$
  
$$\{M_i, M_j\} = \epsilon_{ijk} M_k$$

Angular momentum is given by

$$\mathbf{M} = \mathbf{x} \times \mathbf{p};$$
$$M_i = \epsilon_{ilm} x_l p_m$$

We can also write, for  ${\cal J}_{jk}$ 

$$J_{jk} = p_j x_k - p_k x_j;$$
  
=  $(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) p_l x_m;$   
=  $\epsilon_{ijk} \epsilon_{ilm} p_l x_m.$ 

From the last equality, it is directly written that

$$J_{jk} = -\epsilon_{ijk}M_i = \epsilon_{kji}M_i.$$

Conversely, one can see that

$$\epsilon_{ijk}J_{kj} = \epsilon_{ijk}(p_kx_j - x_kp_j);$$
  
=  $\epsilon_{ijk}x_jp_k + \epsilon_{ikj}x_kp_j;$   
=  $2M_i.$ 

or, equivalently,

$$M_i = \frac{1}{2} \epsilon_{kji} J_{jk}.$$

Using results in problem 7 and the fact that the Poisson bracket is obviously bilinear, this yields

$$\{M_i, x_k\} = \frac{1}{2} \epsilon_{lji} \{J_{jl}, x_k\};$$
  
$$= \frac{1}{2} \epsilon_{lji} (\delta_{lk} x_j - \delta_{jk} x_l);$$
  
$$= \frac{1}{2} (\epsilon_{kji} x_j - \epsilon_{lki} x_l);$$
  
$$= -\frac{1}{2} (\epsilon_{ijk} x_j + \epsilon_{ijk} x_j);$$
  
$$= \epsilon_{jik} x_j;$$

where we have renamed mute variable as we saw fit. Similarly, we get

$$\{M_i, p_k\} = \frac{1}{2} \epsilon_{lji} \{J_{jl}, p_k\};$$
  
$$= \frac{1}{2} \epsilon_{lji} (\delta_{lk} p_j - \delta_{jk} p_l);$$
  
$$= \frac{1}{2} (\epsilon_{kji} p_j - \epsilon_{lki} p_l);$$
  
$$= -\frac{1}{2} (\epsilon_{ijk} p_j + \epsilon_{ijk} p_j);$$
  
$$= \epsilon_{jik} p_j.$$

And finally, we can compute

$$\{M_i, M_k\} = \frac{1}{4} \epsilon_{lji} \epsilon_{mnk} \{J_{jl}, J_{nm}\};$$
  
$$= \frac{1}{4} \epsilon_{lji} \epsilon_{mnk} (\delta_{ln} J_{jm} + \delta_{jm} J_{ln} - \delta_{lm} J_{jn} - \delta_{jn} J_{lm});$$
  
$$= \frac{1}{4} (\epsilon_{lji} \epsilon_{lkm} J_{jm} + \epsilon_{jil} \epsilon_{jnk} J_{ln} - \epsilon_{lji} \epsilon_{lnk} J_{jn} - \epsilon_{jil} \epsilon_{jkm} J_{lm})$$

We can then rename all the mute variables (i.e. everything that is not i or k), in a way that would make us able to sum over J, and rearranging the variables in the Levi-Civita symbol, to get

$$\{M_i, M_k\} = \frac{1}{4} \left( \epsilon_{lji} \epsilon_{lkm} J_{jm} + \epsilon_{lji} \epsilon_{lkm} J_{jm} + \epsilon_{lji} \epsilon_{lkm} J_{jm} + \epsilon_{lji} \epsilon_{lkm} J_{jm} \right);$$
  
=  $\epsilon_{lji} \epsilon_{lkm} J_{jm};$   
=  $\left( \delta_{jk} \delta_{im} - \delta_{jm} \delta_{ik} \right) J_{jm};$   
=  $J_{ki}.$ 

We used in the last computation the fact that  $\delta_{jm}J_{jm} = 0$  since  $J_{\alpha\alpha} = 0 \forall \alpha$ 

# Problem 9

QUESTION: Find the time derivative  $\dot{M}_z$  for the system in exercise Problem 3, by computing  $\{M_z, \mathcal{H}\}$ .

ANSWER : The time derivative of the angular momenta is

$$\frac{dM_z}{dt} = \{M_z, \mathcal{H}\} = x_2 \frac{\partial U}{\partial x^1} - x_1 \frac{\partial U}{\partial x^2}.$$

SOLUTION: First we note that for arbitrary function f(p, q, t) one has

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial \mathcal{H}}{\partial q} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\}.$$
(3.5)

If f does not depend explicitly on time, only on p and q (and only through them on time), which is the case for  $M_i$ , one has

$$\frac{df}{dt} = \{f, \mathcal{H}\}.$$
(3.6)

That is why by computing  $\{f, \mathcal{H}\}$  we compute the time derivative  $\dot{f}$ . Since there is no explicit time dependance in the angular momenta, their time derivative is given by

$$\dot{M}_i = \{M_i, \mathcal{H}\}.$$

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The Hamiltonian in problem 3 was

$$\mathcal{H} = \frac{1}{m} \left[ \frac{1}{2} \left( p_1^2 + p_2^2 \right) - B \left( x_1 p_2 - x_2 p_1 \right) + \frac{1}{2} B^2 \left( x_1^2 + x_2^2 \right) \right] + U(x_1, x_2)$$
  
$$= \frac{1}{2m} \left[ (p_1 + B x_2)^2 + (p_2 - B x_1)^2 \right] + U.$$
(3.7)

Note that the system is two dimensional, hence there is a sense to consider only

$$M_z = x \, p_y - y \, p_x$$

$$\{M_z, \mathcal{H}\} = p_y \frac{\partial \mathcal{H}}{\partial p_x} - p_x \frac{\partial \mathcal{H}}{\partial p_y} - x \frac{\partial \mathcal{H}}{\partial y} + y \frac{\partial \mathcal{H}}{\partial x} = y \frac{\partial U}{\partial x} - x \frac{\partial U}{\partial y} + \frac{1}{m} \left[ p_y (p_x + B y) - p_x (p_y - B x) - B x (p_x + B y) - B y (p_y - B x) \right] = y \frac{\partial U}{\partial x} - x \frac{\partial U}{\partial y}.$$
(3.8)

## Problem 10

QUESTION: Find the time derivatives  $\dot{M}_x, \dot{M}_y, \dot{M}_z$  for the system in Problem 5, by computing  $\{M_i, \mathcal{H}\}$ .

ANSWER:

$$\frac{d\mathbf{M}}{dt} = \{\mathbf{M}, \mathcal{H}\} = -\frac{e}{2c} \frac{1}{m\sqrt{1 + \frac{P^2}{m^2c^2}}} \mathbf{B} \times (\mathbf{r} \times \mathbf{P}).$$

SOLUTION: Solution of this problem depends a lot on how well you think through your computational strategy. Bruteforce approach is likely to generate too big expressions which would be hard to handle.

The first logical step would be to notice that

$$\{\mathcal{H}, \mathbf{M}\} = 2\frac{\partial \mathcal{H}}{\partial P^2} \{\frac{1}{2}P^2, \mathbf{M}\} = \frac{1}{m\sqrt{1 + \frac{P^2}{m^2c^2}}} \{\frac{1}{2}P^2, \mathbf{M}\}.$$
 (3.9)

Hence relativistic nature of the Hamiltonian is not that crucial, it produces only the common prefactor  $\frac{1}{m\sqrt{1+\frac{P^2}{m^2c^2}}}$ , while the non-trivial part of the computation is to find  $\{\frac{1}{2}P^2, \mathbf{M}\}.$ 

Our preparatory work would be to recall several Poisson brackets which were computed in the previous problems or follow from them:

$$\{x_i, M_j\} = \epsilon_{ijk} x_k \,, \quad \{p_i, M_j\} = \epsilon_{ijk} p_k \,, \tag{3.10}$$

$$\{M_i, M_j\} = \epsilon_{ijk} M_k \,. \tag{3.11}$$

$$\{x^2, M_i\} = 0, \quad \{p^2, M_i\} = 0, \tag{3.12}$$

And we also recall relations from the vector algebra

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}), \qquad (3.13)$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}).$$
(3.14)

In fact, both these equations are due to the Plucker identity

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\beta'\gamma'} = \delta_{\beta\beta'}\delta_{\gamma\gamma'} - \delta_{\beta\gamma'}\delta_{\beta'\gamma}. \tag{3.15}$$

If we contract (3.15) with  $A^{\beta}B^{\beta}C^{\beta'}D^{\gamma'}$ , we get (3.13). If we contract (3.15) with  $A^{\gamma}B^{\beta'}C^{\gamma'}$  and leave the index  $\beta$  free, we get (3.14). There is another simple relation we will use:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).$$
(3.16)

In the language of Levi-Civita symbol, it is just the property  $\epsilon_{\alpha\beta\gamma} = \epsilon_{\beta\gamma\alpha} = \epsilon_{\gamma\alpha\beta}$ . Our next step is to expand  $\frac{1}{2}P^2$ . We will use notation  $a \simeq b$  to say that two expressions have the same Poisson bracket with **M**:  $\{a, \mathbf{M}\} = \{b, \mathbf{M}\}$ . In practice, we will throw away  $x^2$  and  $p^2$  terms from the expansion of  $\frac{1}{2}P^2$  because of (3.12).

$$\frac{1}{2}P^{2} = \frac{1}{2}\left(\mathbf{p} - \frac{e}{2c}\mathbf{B}\times\mathbf{r}\right)^{2} = \frac{1}{2}p^{2} - \frac{e}{2c}\mathbf{p}\cdot(\mathbf{B}\times\mathbf{r}) + \frac{1}{2}\left(\frac{e}{2c}\right)^{2}(\mathbf{B}\times\mathbf{r})^{2}$$

$$= \frac{1}{2}p^{2} - \frac{e}{2c}\mathbf{B}\cdot(\mathbf{r}\times\mathbf{p}) + \frac{1}{2}\left(\frac{e}{2c}\right)^{2}\left(B^{2}r^{2} - (\mathbf{B}\cdot\mathbf{r})^{2}\right)$$

$$\simeq -\frac{e}{2c}\mathbf{B}\cdot\mathbf{M} - \frac{1}{2}\left(\frac{e}{2c}\right)^{2}(\mathbf{B}\cdot\mathbf{r})^{2}.$$
(3.17)

$$\{\mathbf{B}\cdot\mathbf{M}, M_j\} = B_i\epsilon_{ijk}M_k = -\epsilon_{jik}B_iM_k, \quad \rightarrow \quad \{\mathbf{B}\cdot\mathbf{M}, \mathbf{M}\} = -\mathbf{B}\times\mathbf{M}. \quad (3.18)$$

$$\{\mathbf{B}\cdot\mathbf{r}, M_j\} = B_i\epsilon_{ijk}r_k = -\epsilon_{jik}B_ir_k, \quad \rightarrow \quad \{\frac{1}{2}(\mathbf{B}\cdot\mathbf{r})^2, \mathbf{M}\} = -\mathbf{B}\times\mathbf{r}(\mathbf{B}\cdot\mathbf{r}).$$
(3.19)

Now we note that

$$\mathbf{B} \times \mathbf{M} = \mathbf{B} \times (\mathbf{r} \times \mathbf{p}) = \mathbf{r} (\mathbf{B} \cdot \mathbf{p}) - \mathbf{p} (\mathbf{B} \cdot \mathbf{r}) = \mathbf{r} (\mathbf{B} \cdot \mathbf{P}) - \mathbf{p} (\mathbf{B} \cdot \mathbf{r}).$$
(3.20)

We are now ready to compute

$$\{\frac{1}{2}P^{2}, \mathbf{M}\} = \frac{e}{2c}\mathbf{B} \times \mathbf{M} + \left(\frac{e}{2c}\right)^{2}\mathbf{B} \times \mathbf{r}(\mathbf{B} \cdot \mathbf{r}) =$$
$$= \frac{e}{2c}\mathbf{r}(\mathbf{B} \cdot \mathbf{P}) - \frac{e}{2c}\mathbf{p}(\mathbf{B} \cdot \mathbf{r}) + \left(\frac{e}{2c}\right)^{2}\mathbf{B} \times \mathbf{r}(\mathbf{B} \cdot \mathbf{r})$$
$$= \frac{e}{2c}(\mathbf{r}(\mathbf{B} \cdot \mathbf{P}) - \mathbf{P}(\mathbf{B} \cdot \mathbf{r})) = \frac{e}{2c}\mathbf{B} \times (\mathbf{r} \times \mathbf{P}).$$
(3.21)

And finally

$$\{\mathbf{M}, \mathcal{H}\} = -\frac{e}{2c} \frac{1}{m\sqrt{1 + \frac{P^2}{m^2 c^2}}} \mathbf{B} \times (\mathbf{r} \times \mathbf{P}).$$
(3.22)

DISCUSSION: The derivation was quite lengthy. We can do it differently and faster by computing directly  $\dot{\mathbf{M}}$  since we already know  $\dot{\mathbf{r}}$  and  $\dot{\mathbf{p}}$  from Problem 5:

$$\dot{\mathbf{M}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \frac{1}{m} \frac{1}{\sqrt{1 + \frac{P^2}{m^2 c^2}}} \left( \mathbf{P} \times \mathbf{p} - \frac{e}{2c} \mathbf{r} \times (\mathbf{B} \times \mathbf{P}) \right) =$$
$$= \frac{e}{2c} \frac{1}{m} \frac{1}{\sqrt{1 + \frac{P^2}{m^2 c^2}}} \left( \mathbf{P} \times (\mathbf{B} \times \mathbf{r}) - \mathbf{r} \times (\mathbf{B} \times \mathbf{P}) \right), \qquad (3.23)$$

and after a simple algebra we again get (3.22).