HW1: Basics of Hamiltonian mechanics

If you find misprints, have any questions, find some task difficult and want a hint, contact me by email vel145@gmail.com.

A bit of differential forms

The first two exercises is a small check that you followed lecture and tutorial from the first week.

1. Integrate $(x^3 - 2\,x^2\,y + x\,y^2 - 2\,y^3)\,dx - 2\,x\,dy$ over a circle of unit radius in a counterclockwise direction. Explain the obtained result.

2. If you have a force $\mathbf{F}$ acting on a particle, you can define the differential form $F_i dx^i$ (summation over i is assumed). Then the work over a path γ is given by $\int_\gamma F_i dx^i$.

   Translate the notion of "conservative" and "non-conservative" force to the language of differential forms.

Hamiltonian and Hamilton's equations of motion

All necessary definitions are given on wikipedia, article Hamiltonian mechanics, and in Landau-Lifshitz book, chapter VII, section §40.

3. Find the Hamiltonian and Hamilton's equations of motion for a system with the following Lagrangian

$$\mathcal{L} = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + B(x_1\dot{x}_2 - x_2\dot{x}_1) + U(x_1, x_2)\,. \tag{1}$$

4. Rewrite the Lagrangian from the previous exercise in polar coordinates and find the Hamiltonian and Hamilton's equations of motion in terms of generalised coordinates and momenta: $\{r, \phi, p_r, p_\phi\}$.

5. Find the Hamiltonian and Hamilton's equations of motion for a relativistic particle of charge e which is moving in the constant magnetic field $\mathbf{B}$. The lagrangian is given by:

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{2c}\epsilon_{\alpha\beta\gamma}B_\alpha x_\beta v_\gamma\,, \tag{2}$$

   where $v_\alpha = \dot{x}_\alpha$ is the velocity vector and v is its absolute value. $\epsilon_{\alpha\beta\gamma}$ is the fully antisymmetric tensor with $\epsilon_{123} = 1$.

   Note: In the expression $\epsilon_{\alpha\beta\gamma}B_\alpha x_\beta v_\gamma$ summation over the repeated index is assumed.

   Note: In this task, m, c, e, B_1, B_2, B_3 should be considered as numerical constants (they correspond, respectively, to mass of the particle, speed of light, electric charge, and components of magnetic field. The Legendre transform should be performed with respect to $\dot{x}_1, \dot{x}_2, \dot{x}_3$.

Poisson brackets

All necessary definitions are given on wikipedia, article Poisson bracket (not necessary to understand the coordinate-free language there). Also Landau-Lifshitz book, chapter VII, section §42.

6. Prove the Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

7. The Hamiltonian of a free particle in D dimensions is given by $\mathcal{H} = \sum_{i=1}^{\mathrm{D}} \frac{p_i^2}{2m}$. Consider the quantity $J_{ij} = p_i x_j - p_j x_i$. Compute the following poisson brackets:

$$\{J_{ij}, p_k\}, \quad \{J_{ij}, x_k\}, \quad \{J_{ij}, J_{kl}\}, \tag{3}$$

in the case of $\{J_{ij}, J_{kl}\}$, express the answer as a linear combination of J's only.

What is the value of $\{J_{14}, J_{13}\}$?

Prove that J_{ij} is the conserved quantity by computing $\{J_{ij}, \mathcal{H}\}$.

In the questions below, the angular momentum vector $\mathbf{M} = \{M_1, M_2, M_3\}$ is defined as vector product $\mathbf{M} = \mathbf{x} \times \mathbf{p}$. Note: it is not always the same as $\mathbf{x} \times (m\dot{\mathbf{x}})$.

8. For the case of free particle in D=3, express angular momenta M_i in terms of J_{jk} using ϵ_{ijk}. And vice versa, express J's in terms of M's. Find the Poisson brackets

$$\{M_i, p_k\}, \quad \{M_i, x_k\}, \quad \{M_i, M_k\}. \tag{4}$$

If possible, express the answer in terms of M again.

9. Find the time derivative $\dot{M}_z$ for the system in exercise 3, by computing $\{M_i, \mathcal{H}\}$.

10. Find the time derivatives $\dot{M}_x, \dot{M}_y, \dot{M}_z$ for the system in exercise 5, by computing $\{M_i, \mathcal{H}\}$.

Note: Sign convention for Poisson brackets may differ in different sources. We use the one which is published currently on Wikipedia. Be careful with this sign when you compute time derivatives!