## Sample of exam for MA2342 - Avanced Classical Mechanics II (Spring 2014)

All questions have equal weight.
Credit will be given for the best 3 questions
Note: Presence of some question in this sample does not mean you will not get the same question in the real exam. It does not mean either that you will get it.

## 1. Hamiltonian mechanics

(a) [5 points] Rewrite the Lagrangian of 3-dimensional free particle of mass $m$ in the spherical coordinates. Put then $r(t)=R$, where $R$ is some constant, and $\dot{r}=0$. This will be a Lagrangian describing a particle moving on a sphere. It should have only 2 dynamical degrees of freedom: $\theta$ and $\phi$.
(b) [5 points] Find conjugated momenta $p_{\phi}, p_{\theta}$ and write down the Hamiltonian and Hamiltonian equations of motion.
(c) [5 points] Rewrite the components of the angular momentum $\mathbf{M}=m \mathbf{r} \times \dot{\mathbf{r}}$ in terms of $\theta, \phi, p_{\theta}, p_{\phi}$
(d) [5 points] Compute the following Poisson brackets: $\left\{M_{i}, \mathcal{H}\right\}$ and $\left\{M_{i}, M_{j}\right\}$, express the answer in terms of $M_{x}, M_{y}, M_{z}$ again.

## 2. Hamilton-Jacobi equation

(a) [5 points] Give a derivation of the time-dependent Hamilton-Jacobi equation. Derive the time-independent equation from it.
(b) [6 points] A solution of a Hamilton-Jacobi equation is $S=W-E t$, with

$$
W=x \sqrt{2 E-\frac{1}{x^{2}}}+\arctan \left(\frac{1}{x \sqrt{2 E-\frac{1}{x^{2}}}}\right)
$$

Find the Hamiltonian of a system which has such solution
(c) [6 points] Departing from the above-given $S(x, E, t)$, find the trajectory of the system in the phase space. The answer should be in the form $p=p(t, E), x=x(t, E)$.
(d) [3 points] What is the closest position $x$ of the particle of energy $E$ to the origin $x=0$ in this system?

## 3. Special relativity

(a) [6 points] Describe/classify all possible Lorentz transformations in $1+3$ dimensions.
(b) [6 points] In $1+1$ dimensional space a function $\Phi\left(p 1^{\mu}, p 2^{\nu}\right)$ is a Lorentz scalar which depends on two 2-momenta: $p 1$ and $p 2$. Show that $\Phi$ depends actually only on the difference of rapidities $\theta_{1}-\theta_{2}$. Rapidity $\theta_{1}$ of $p 1$ is defined $p 1^{0}=m_{1} c \cosh \theta_{1}$ and rapidity $\theta_{2}$ of $p 1$ is defined $p 2^{0}=m_{2} c \cosh \theta_{2}$.
(c) [8 points] Sketch the twin paradox. The full score will be given if you explain in detail where is the logical mistake hidden and how is it resolved in the reference frame of the traveling twin.
4. Particle in Electromagnetic field Particle in the Electromagnetic field is in total described by the action:

$$
\begin{equation*}
S=-m c \int \sqrt{\eta_{\mu \nu} \frac{d x^{\mu}}{d \theta} \frac{d x^{\nu}}{d \theta}} d \theta-\frac{e}{c} \int A_{\mu}(x) d x^{\mu}-\frac{1}{16 \pi c} \int d^{4} x F_{\mu \nu} F^{\mu \nu} \tag{1}
\end{equation*}
$$

where $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$
(a) [3 points] Show that $\partial_{\mu} F_{\nu \lambda}+\partial_{\nu} F_{\lambda \mu}+\partial_{\lambda} F_{\mu \nu}=0$.
(b) [6 points] Write down equations of motion for a particle and the Maxwell equations
(c) [5 points] Electromagnetic tensor $F_{\mu \nu}$ is known to be a covariant rank 2 object. It has explicitly the following components:

$$
F_{\mu \nu}=\left(\begin{array}{cccc}
0 & E^{x} & E^{y} & E^{z}  \tag{2}\\
-E^{x} & 0 & -B^{z} & B^{y} \\
-E^{y} & B^{z} & 0 & -B^{x} \\
-E^{z} & -B^{y} & B^{x} & 0
\end{array}\right) .
$$

Based on this information, decide which of the following combinations are invariant under Lorentz transformations: $\mathbf{E} \cdot \mathbf{B}, \mathbf{E}^{2}+\mathbf{B}^{2}, \mathbf{E}^{2}-\mathbf{B}^{2},|\mathbf{E} \times \mathbf{B}|$ ?
Hint: Consider $\operatorname{det} F$ and $F_{\mu \nu} F^{\mu \nu}$.
(d) [6 points]. Using (2) and choosing $\theta=c t$, rewrite equations from question (b) in 3-dimensional notations.

