Proof of Gordon Decomposition identity

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We are given

$$\sigma^{\mu\nu} = \frac{i}{2}\,[\,\gamma^{\mu}, \gamma^{\nu}\,]$$

we know

$$\{\,\gamma^{\mu}, \gamma^{\nu}\,\} = \gamma^{\mu}\,\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\,\eta^{\mu\nu}$$

so

$$[\,\gamma^{\mu}, \gamma^{\nu}\,] = \gamma^{\mu}\,\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu} = \gamma^{\mu}\,\gamma^{\nu} - (2\,\eta^{\mu\nu} - \gamma^{\mu}\,\gamma^{\nu}) = 2\,\gamma^{\mu}\,\gamma^{\nu} - 2\,\eta^{\mu\nu}$$

Hence

$$i\,\sigma^{\mu\nu} = -\,(\,\gamma^{\mu}\,\gamma^{\nu} - \eta^{\mu\nu}\,) = \eta^{\mu\nu} - \gamma^{\mu}\,\gamma^{\nu}$$

But equivalently

$$i\,\sigma^{\mu\nu} = \eta^{\mu\nu} - (2\,\eta^{\mu\nu} - \gamma^{\nu}\,\gamma^{\mu}\,) = \gamma^{\nu}\,\gamma^{\mu} - \eta^{\mu\nu}$$

So we calculate

$$\begin{aligned}
\bar{u}(p')\,i\,\sigma^{\mu\nu}\,(\,p'_{\nu} - p_{\nu}\,)\;u(p) &= \bar{u}(p')\,[(\gamma^{\nu}\,\gamma^{\mu} - \eta^{\mu\nu})\,p'_{\nu} - (\eta^{\mu\nu} - \gamma^{\mu}\,\gamma^{\nu}\,)\,p_{\nu}]\,u(p) \\
&= \bar{u}(p')\,[\gamma^{\nu}p'_{\nu}\gamma^{\mu} - p'^{\mu} - p^{\mu} + \gamma^{\mu}\gamma^{\nu}p_{\nu}]\,u(p) \\
&= \bar{u}(p')\left[\gamma\cdot p'\,\gamma^{\mu} - (\,p' + p\,)^{\mu} + \gamma^{\mu}\,\gamma\cdot p\right]u(p)
\end{aligned}$$

Now simplify by using the Dirac equation and conjugate

$$\begin{aligned}
(\,\gamma\cdot p - M\,)\,u(p) &= 0 \quad\Longrightarrow\quad \gamma\cdot p\;u(p) = M\,u(p) \\
\bar{u}(p')\,(\,\gamma\cdot p' - M\,) &= 0 \quad\Longrightarrow\quad \bar{u}(p')\;\gamma\cdot p' = \bar{u}(p')\,M
\end{aligned}$$

on the first and last terms in our above expression to obtain

$$\begin{aligned}
\bar{u}(p')\,i\,\sigma^{\mu\nu}\,(\,p'_{\,\nu} - p_{\nu}\,)\;u(p) &= \bar{u}(p')\left[\,M\,\gamma^{\mu} - (\,p' + p\,)^{\mu} + \gamma^{\mu}\,M\,\right]u(p) \\
&= \bar{u}(p')\left[\,2\,M\,\gamma^{\mu} - (\,p' + p\,)^{\mu}\,\right]u(p)
\end{aligned}$$

Thus we have shown

$$\bar{u}(p')\,i\,\sigma^{\mu\nu}\,(\,p'_{\,\nu} - p_{\nu}\,)\;u(p) = \bar{u}(p')\left[\,2\,M\,\gamma^{\mu} - (\,p' + p\,)^{\mu}\,\right]u(p)$$

Rearranging gives the general form of the **Gordon Decomposition identity**

$$\boxed{\bar{u}(p')\,\gamma^{\mu}\,u(p) = \bar{u}(p')\left[\frac{(\,p' + p\,)^{\mu}}{2\,M} + \frac{i\,\sigma^{\mu\nu}\,(\,p'_{\nu} - p_{\nu}\,)}{2\,M}\right]u(p)}$$