Assignment 8 MA1124

Q 52, 53, 54 Attached.

4. Also try to use the l.u.b. axiom to prove $[a, b]$ is compact.

Hint $[a, c]$ can be covered by one O_γ for some $c > a$.

51. Prove: If $a_n \to a$ and $b_n \to b$ where $b_n \neq 0$ and $b \neq 0$, then the sequence $\langle a_n/b_n \rangle$ converges to a/b.

52. Prove: If the sequence $\langle a_n \rangle$ converges to b, then every subsequence $\langle a_{i_n} \rangle$ of $\langle a_n \rangle$ also converges to b.

53. Prove: If the sequence $\langle a_n \rangle$ converges to b, then either the range $\{a_n\}$ of the sequence $\langle a_n \rangle$ is finite, or b is an accumulation point of the range $\{a_n\}$.

54. Prove: If the sequence $\langle a_n \rangle$ of distinct elements is bounded and the range $\{a_n\}$ of $\langle a_n \rangle$ has exactly one limit point b, then the sequence $\langle a_n \rangle$ converges to b.
(*Remark:* The sequence $\langle 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots \rangle$ shows that the condition of boundedness cannot be removed from this theorem.)

CONTINUITY

55. Prove: A function $f: \mathbf{R} \to \mathbf{R}$ is continuous at $a \in \mathbf{R}$ if and only if for every sequence $\langle a_n \rangle$ converging to a, the sequence $\langle f(a_n) \rangle$ converges to $f(a)$.

56. Prove: Let the function $f: \mathbf{R} \to \mathbf{R}$ be continuous at $p \in \mathbf{R}$. Then there exists an open interval S containing p such that f is bounded on the open interval S.

57. Give an example of a function $f: \mathbf{R} \to \mathbf{R}$ which is continuous at every point in the open interval $S = (0, 1)$ but which is not bounded on the open interval S.

58. Prove: Let $f: \mathbf{R} \to \mathbf{R}$ be continuous at every point in a closed interval $A = [a, b]$. Then f is bounded on A. (*Remark:* By the preceding problem, this theorem is not true if A is not closed.)

59. Prove: Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be continuous. Then the sum $(f+g): \mathbf{R} \to \mathbf{R}$ is continuous, where $f+g$ is defined by $(f+g)(x) \equiv f(x) + g(x)$.

60. Prove: Let $f: \mathbf{R} \to \mathbf{R}$ be continuous, and let k be any real number. Then the function $(kf): \mathbf{R} \to \mathbf{R}$ is continuous, where kf is defined by $(kf)(x) \equiv k(f(x))$.

61. Prove: Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be continuous. Then $\{x \in \mathbf{R} : f(x) = g(x)\}$ is a closed set.

62. Prove: The projection $\pi_x: \mathbf{R}^2 \to \mathbf{R}$ is continuous where π_x is defined by $\pi_x(\langle a, b \rangle) = a$.

63. Consider the functions $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \qquad g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove g is continuous at 0 but f is not continuous at 0.

64. Recall that every rational number $q \in \mathbf{Q}$ can be written uniquely in the form $q = a/b$ where $a \in \mathbf{Z}$, $b \in \mathbf{N}$, and a and b are relatively prime. Consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by

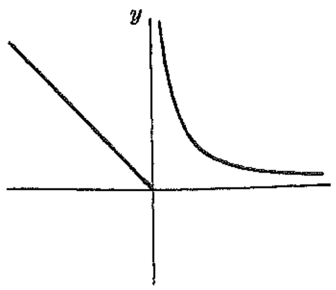
$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/b & \text{if } x \text{ is rational and } x = a/b \text{ as above} \end{cases}$$

Prove that f is continuous at every irrational point, but f is discontinuous at every rational point.

Answers to Supplementary Problems

57. Consider the function

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 1/x & \text{if } x > 0 \end{cases}$$



The function f is continuous at every point in $\mathbf{R}$ except at 0 as indicated in the adjacent graph of f. Hence f is continuous at every point in the open interval $(0, 1)$. But f is not bounded on $(0, 1)$.

58. *Hint.* Use the result stated in Problem 56 and the Heine-Borel Theorem.