Assignment 6 Due Thurs 10^{th} March

1. Prove $\lim_{x \to a} f(x) = L \iff \forall x_n \to a$, $f(x_n) \to L$

Do #35, 40, 45 from page 64.

We claim that $f(p) = 0$. If $f(p) < 0$, then, by the preceding problem, there is an open interval $(p-\delta, p+\delta)$ in which f is negative, i.e.,

$$(p-\delta, p+\delta) \subset A$$

So p cannot be an upper bound for A. On the other hand, if $f(p) > 0$, then there exists an interval $(p-\delta, p+\delta)$ in which f is positive; so

$$(p-\delta, p+\delta) \cap A \;=\; \emptyset$$

which implies that p cannot be a least upper bound for A. Thus $f(p)$ can only be zero, i.e. $f(p) = 0$.

Remark. The theorem is also true and proved similarly in the case $f(b) < 0 < f(a)$.

37. Prove Theorem (Weierstrass) 4.9: Let $f: \mathbf{R} \to \mathbf{R}$ be continuous on a closed interval $[a, b]$. Then the function assumes every value between $f(a)$ and $f(b)$.

Solution:

Suppose $f(a) < f(b)$ and let y_0 be a real number such that $f(a) < y_0 < f(b)$. We want to prove that there is a point p such that $f(p) = y_0$. Consider the function $g(x) = f(x) - y_0$ which is also continuous. Observe that $g(a) < 0 < g(b)$.

By the preceding problem, there exists a point p such that $g(p) = f(p) - y_0 = 0$. Hence $f(p) = y_0$.

The case when $f(b) < f(a)$ is proved similarly.

Supplementary Problems

OPEN SETS, CLOSED SETS, ACCUMULATION POINTS

38. Prove: If A is a finite subset of $\mathbf{R}$, then the derived set A' of A is empty, i.e. $A' = \emptyset$.

39. Prove: Every finite subset of $\mathbf{R}$ is closed.

40. Prove: If $A \subset B$, then $A' \subset B'$.

41. Prove: A subset B of $\mathbf{R}^2$ is closed if and only if $d(p, B) = 0$ implies $p \in B$, where $d(p, B) = \inf\{d(p, q) : q \in B\}$.

42. Prove: $A \cup A'$ is closed for any set A.

43. Prove: $A \cup A'$ is the smallest closed set containing A, i.e. if F is closed and $A \subset F \subset A \cup A'$ then $F = A \cup A'$.

44. Prove: The set of interior points of any set A, written $\operatorname{int}(A)$, is an open set.

45. Prove: The set of interior points of A is the largest open set contained in A, i.e. if G is open and $\operatorname{int}(A) \subset G \subset A$, then $\operatorname{int}(A) = G$.

46. Prove: The only subsets of $\mathbf{R}$ which are both open and closed are $\emptyset$ and $\mathbf{R}$.

SEQUENCES

47. Prove: If the sequence $\langle a_n \rangle$ converges to $b \in \mathbf{R}$, then the sequence $\langle |a_n - b| \rangle$ converges to 0.

48. Prove: If the sequence $\langle a_n \rangle$ converges to 0, and the sequence $\langle b_n \rangle$ is bounded, then the sequence $\langle a_n b_n \rangle$ also converges to 0.

49. Prove: If $a_n \to a$ and $b_n \to b$, then the sequence $\langle a_n + b_n \rangle$ converges to $a + b$.

50. Prove: If $a_n \to a$ and $b_n \to b$, then the sequence $\langle a_n b_n \rangle$ converges to ab.