MA1124 Assignment5
[due Monday 16 February, 2015]

1. p233, 22,30 attached.
2. p64, 44,45,47,48 attached.
3. If A_α are open, what about finite unions? countable unions? All unions? Same for intersections?
4. If A_α are closed, same questions.
5. Prove that A is open iff $\forall x \in A, x_n \to x,$, implies the sequence x_n is ultimately in A, ie $\exists$ N such that $n \geq N \to x_n \in$ A.

Solution:

Suppose p_1 and p_2 belong to every interval. If $p_1 \neq p_2$, then $|p_1 - p_2| = \delta > 0$. Since $\lim_{n\to\infty} (b_n - a_n) = 0$, there exists an interval $I_{n_0} = [a_{n_0}, b_{n_0}]$ such that the length of I_{n_0} is less than the distance $|p_1 - p_2| = \delta$ between p_1 and p_2. Accordingly, p_1 and p_2 cannot both belong to I_{n_0}, a contradiction. Thus $p_1 = p_2$, i.e. only one point can belong to every interval.

Supplementary Problems

FIELD AXIOMS

20. Show that the Right Distributive Law $[D_2]$ is a consequence of the Left Distributive Law $[D_1]$ and the Commutative Law $[M_5]$.

21. Show that the set **Q** of rational numbers under addition and multiplication is a field.

22. Show that the following set A of real numbers under addition and multiplication is a field:
$$A = \{a + b\sqrt{2} : a, b \text{ rational}\}$$

23. Show that the set $A = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers under addition and multiplication satisfies all the axioms of a field except $[M_3]$, $[M_4]$ and $[M_5]$, that is, is a ring.

INEQUALITIES AND POSITIVE NUMBERS

24. Rewrite so that x is alone between the inequality signs:
(i) $4 < -2x < 10$, (ii) $-1 < 2x - 3 < 5$, (iii) $-3 < 5 - 2x < 7$.

25. Prove: The product of any two negative numbers is positive.

26. Prove Theorem A.2(iii): If $a < b$, then $a + c < b + c$.

27. Prove Theorem A.2(iv): If $a < b$ and c is positive, then $ac < bc$.

28. Prove Corollary A.3: The set **R** of real numbers is totally ordered by the relation $a \leq b$.

29. Prove: If $a < b$ and c is positive, then: (i) $\frac{a}{c} < \frac{b}{c}$, (ii) $\frac{c}{b} < \frac{c}{a}$.

30. Prove: $\sqrt{ab} \leq (a+b)/2$. More generally, prove $\sqrt[n]{a_1 a_2 \cdots a_n} \leq (a_1 + a_2 + \cdots + a_n)/n$.

31. Prove: Let a and b be real numbers such that $a < b + \epsilon$ for every $\epsilon > 0$. Then $a \leq b$.

32. Determine all real values of x such that: (i) $x^3 + x^2 - 6x > 0$, (ii) $(x-1)(x+3)^2 \leq 0$.

ABSOLUTE VALUES

33. Evaluate: (i) $|-2| + |1 - 4|$, (ii) $|3 - 8| - |1 - 9|$, (iii) $||-4| - |2 - 7||$.

34. Rewrite, using the absolute value sign: (i) $-3 < x < 9$, (ii) $2 \leq x \leq 8$, (iii) $-7 < x < -1$.

35. Prove: (i) $|-a| = |a|$, (ii) $a^2 = |a|^2$, (iii) $|a| = \sqrt{a^2}$, (iv) $|x| < a$ iff $-a < x < a$.

We claim that $f(p) = 0$. If $f(p) < 0$, then, by the preceding problem, there is an open interval $(p-\delta,\ p+\delta)$ in which f is negative, i.e.,

$$(p-\delta,\ p+\delta) \subset A$$

So p cannot be an upper bound for A. On the other hand, if $f(p) > 0$, then there exists an interval $(p-\delta,\ p+\delta)$ in which f is positive; so

$$(p-\delta,\ p+\delta) \cap A = \emptyset$$

which implies that p cannot be a least upper bound for A. Thus $f(p)$ can only be zero, i.e. $f(p) = 0$.

Remark. The theorem is also true and proved similarly in the case $f(b) < 0 < f(a)$.

37. Prove Theorem (Weierstrass) 4.9: Let $f: \mathbf{R} \to \mathbf{R}$ be continuous on a closed interval $[a, b]$. Then the function assumes every value between $f(a)$ and $f(b)$.

Solution:

Suppose $f(a) < f(b)$ and let y_0 be a real number such that $f(a) < y_0 < f(b)$. We want to prove that there is a point p such that $f(p) = y_0$. Consider the function $g(x) = f(x) - y_0$ which is also continuous. Observe that $g(a) < 0 < g(b)$.

By the preceding problem, there exists a point p such that $g(p) = f(p) - y_0 = 0$. Hence $f(p) = y_0$.

The case when $f(b) < f(a)$ is proved similarly.

Supplementary Problems

OPEN SETS, CLOSED SETS, ACCUMULATION POINTS

38. Prove: If A is a finite subset of $\mathbf{R}$, then the derived set A' of A is empty, i.e. $A' = \emptyset$.

39. Prove: Every finite subset of $\mathbf{R}$ is closed.

40. Prove: If $A \subset B$, then $A' \subset B'$.

41. Prove: A subset B of $\mathbf{R}^2$ is closed if and only if $d(p, B) = 0$ implies $p \in B$, where $d(p, B) = \inf\{d(p, q) : q \in B\}$.

42. Prove: $A \cup A'$ is closed for any set A.

43. Prove: $A \cup A'$ is the smallest closed set containing A, i.e. if F is closed and $A \subset F \subset A \cup A'$ then $F = A \cup A'$.

44. Prove: The set of interior points of any set A, written $\operatorname{int}(A)$, is an open set.

45. Prove: The set of interior points of A is the largest open set contained in A, i.e. if G is open and $\operatorname{int}(A) \subset G \subset A$, then $\operatorname{int}(A) = G$.

46. Prove: The only subsets of $\mathbf{R}$ which are both open and closed are $\emptyset$ and $\mathbf{R}$.

SEQUENCES

47. Prove: If the sequence $\langle a_n \rangle$ converges to $b \in \mathbf{R}$, then the sequence $\langle |a_n - b| \rangle$ converges to 0.

48. Prove: If the sequence $\langle a_n \rangle$ converges to 0, and the sequence $\langle b_n \rangle$ is bounded, then the sequence $\langle a_n b_n \rangle$ also converges to 0.

49. Prove: If $a_n \to a$ and $b_n \to b$, then the sequence $\langle a_n + b_n \rangle$ converges to $a + b$.

50. Prove: If $a_n \to a$ and $b_n \to b$, then the sequence $\langle a_n b_n \rangle$ converges to ab.