Assignment 5 MA1124 Due Wednesday24th.

1. On page 233 of the text, attached 24,27,31.
2. On page 234, attached, 37,38,42

Solution:

Suppose p_1 and p_2 belong to every interval. If $p_1 \neq p_2$, then $|p_1 - p_2| = \delta > 0$. Since $\lim_{n\to\infty} (b_n - a_n) = 0$, there exists an interval $I_{n_0} = [a_{n_0}, b_{n_0}]$ such that the length of I_{n_0} is less than the distance $|p_1 - p_2| = \delta$ between p_1 and p_2. Accordingly, p_1 and p_2 cannot both belong to I_{n_0}, a contradiction. Thus $p_1 = p_2$, i.e. only one point can belong to every interval.

Supplementary Problems

FIELD AXIOMS

20. Show that the Right Distributive Law [D_2] is a consequence of the Left Distributive Law [D_1] and the Commutative Law [M_5].

21. Show that the set **Q** of rational numbers under addition and multiplication is a field.

22. Show that the following set A of real numbers under addition and multiplication is a field:
$$A = \{a + b\sqrt{2} : a, b \text{ rational}\}$$

23. Show that the set $A = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers under addition and multiplication satisfies all the axioms of a field except [M_3], [M_4] and [M_5], that is, is a ring.

INEQUALITIES AND POSITIVE NUMBERS

24. Rewrite so that x is alone between the inequality signs:
(i) $4 < -2x < 10$, (ii) $-1 < 2x - 3 < 5$, (iii) $-3 < 5 - 2x < 7$.

25. Prove: The product of any two negative numbers is positive.

26. Prove Theorem A.2(iii): If $a < b$, then $a + c < b + c$.

27. Prove Theorem A.2(iv): If $a < b$ and c is positive, then $ac < bc$.

28. Prove Corollary A.3: The set **R** of real numbers is totally ordered by the relation $a \leq b$.

29. Prove: If $a < b$ and c is positive, then: (i) $\frac{a}{c} < \frac{b}{c}$, (ii) $\frac{c}{b} < \frac{c}{a}$.

30. Prove: $\sqrt{ab} \leq (a + b)/2$. More generally, prove $\sqrt[n]{a_1 a_2 \cdots a_n} \leq (a_1 + a_2 + \cdots + a_n)/n$.

31. Prove: Let a and b be real numbers such that $a < b + \epsilon$ for every $\epsilon > 0$. Then $a \leq b$.

32. Determine all real values of x such that: (i) $x^3 + x^2 - 6x > 0$, (ii) $(x-1)(x+3)^2 \leq 0$.

ABSOLUTE VALUES

33. Evaluate: (i) $|-2| + |1 - 4|$, (ii) $|3 - 8| - |1 - 9|$, (iii) $||-4| - |2 - 7||$.

34. Rewrite, using the absolute value sign: (i) $-3 < x < 9$, (ii) $2 \leq x \leq 8$, (iii) $-7 < x < -1$.

35. Prove: (i) $|-a| = |a|$, (ii) $a^2 = |a|^2$, (iii) $|a| = \sqrt{a^2}$, (iv) $|x| < a$ iff $-a < x < a$.

36. Prove Proposition A.4(ii): $|ab| = |a|\,|b|$.

37. Prove Proposition A.4(iv): $\big|\,|a| - |b|\,\big| \leq |a - b|$.

LEAST UPPER BOUND AXIOM

38. Prove: Let A be a set of real numbers bounded from below. Then A has a greatest lower bound, i.e. $\inf(A)$ exists.

39. Prove: (i) Let $x \in \mathbf{R}$ such that $x^2 < 2$; then $\exists\, n \in \mathbf{N}$ such that $(x + 1/n)^2 < 2$.
(ii) Let $x \in \mathbf{R}$ such that $x^2 > 2$; then $\exists\, n \in \mathbf{N}$ such that $(x - 1/n)^2 > 2$.

40. Prove: There exists a real number $a \in \mathbf{R}$ such that $a^2 = 2$.

41. Prove: Between any two positive real numbers lies a number of the form r^2, where r is rational.

42. Prove: Between any two real numbers there is an irrational number.

Answers to Supplementary Problems

24. (i) $-5 < x < -2$ (ii) $1 < x < 4$ (iii) $-1 < x < 4$

32. (i) $-3 < x < 0$ or $x > 2$, i.e. $x \in (-3, 0) \cup (2, \infty)$ (ii) $x \leq 1$

33. (i) 5 (ii) -3 (iii) 1

34. (i) $|x - 3| < 6$ (ii) $|x - 5| \leq 3$ (iii) $|x + 4| < 3$