MA 1124 Assignment 4 2016

Due Wed 17th.

Q1 If $X \sim Y$ prove $\mathcal{P}(X) \sim \mathcal{P}(Y)$.
What about the converse?

Questions on attached pages

p 29 # 39
p 30 # 42, 43, 44, 50.
p 45 # 29, 30.

25. Let $k: X \to X$ be a constant function. Prove that for any function $f: X \to X$, $k \circ f = k$. What can be said about $f \circ k$?

26. Consider the function $f(x) = x$ where $x \in \mathbf{R}$, $x \geq 0$. State whether or not each of the following functions is an extension of f.

(i) $g_1(x) = |x|$ for all $x \in \mathbf{R}$

(ii) $g_2(x) = x$ where $x \in [-1, 1]$

(iii) $g_3(x) = (x + |x|)/2$ for all $x \in \mathbf{R}$

(iv) $1_{\mathbf{R}}: \mathbf{R} \to \mathbf{R}$

27. Let $A \subset X$ and let $f: X \to Y$. The *inclusion function* j from A into X, denoted by $j: A \subset X$, is defined by $j(a) = a$ for all $a \in A$. Show that $f \mid A$, the restriction of f to A, equals the composition $f \circ j$, i.e. $f \mid A = f \circ j$.

ONE-ONE, ONTO, INVERSE AND IDENTITY FUNCTIONS

28. Prove: For any function $f: A \to B$, $f \circ 1_A = f = 1_B \circ f$.

29. Prove: If $f: A \to B$ is both one-one and onto, then $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

30. Prove: If $f: A \to B$ and $g: B \to A$ satisfy $g \circ f = 1_A$, then f is one-one and g is onto.

31. Prove Proposition 2.1: Let $f: A \to B$ and $g: B \to A$ satisfy $g \circ f = 1_A$ and $f \circ g = 1_B$. Then $f^{-1}: B \to A$ exists and $g = f^{-1}$.

32. Under what conditions will the projection $\pi_{i_0}: \prod \{A_i : i \in I\} \to A_{i_0}$ be one-to-one?

33. Let $f: (-1, 1) \to \mathbf{R}$ be defined by $f(x) = x/(1 - |x|)$. Prove that f is both one-one and onto.

34. Let R be an equivalence relation in a non-empty set A. The natural function η from A into the quotient set A/R is defined by $\eta(a) = [a]$, the equivalence class of a. Prove that η is an onto function.

35. Let $f: A \to B$. The relation R in A defined by $a \, R \, a'$ iff $f(a) = f(a')$ is an equivalence relation. Let $\hat{f}$ denote the correspondence from the quotient set A/R into the range $f[A]$ of f by $\hat{f}: [a] \to f(a)$.

(i) Prove that $\hat{f}: A/R \to f[A]$ is a function which is both one-one and onto.

(ii) Prove that $f = j \circ \hat{f} \circ \eta$, where $\eta: A \to A/R$ is the natural function and $j: f[A] \subset B$ is the inclusion function.

$$A \xrightarrow{\eta} A/R \xrightarrow{\hat{f}} f[A] \xrightarrow{j} B$$

INDEXED SETS AND GENERALIZED OPERATIONS

36. Let $A_n = \{x : x \text{ is a multiple of } n\} = \{n, 2n, 3n, \ldots\}$, where $n \in \mathbf{N}$, the positive integers. Find: (i) $A_2 \cap A_7$; (ii) $A_6 \cap A_8$; (iii) $A_3 \cup A_{12}$; (iv) $A_3 \cap A_{12}$; (v) $A_s \cup A_{st}$, where $s, t \in \mathbf{N}$; (vi) $A_s \cap A_{st}$, where $s, t \in \mathbf{N}$. (vii) Prove: If $J \subset \mathbf{N}$ is infinite, then $\cap \{A_i : i \in J\} = \emptyset$.

37. Let $B_i = (i, i+1]$, an open-closed interval, where $i \in \mathbf{Z}$, the integers. Find:

(i) $B_4 \cup B_5$

(ii) $B_6 \cap B_7$

(iii) $\cup_{i=4}^{20} B_i$

(iv) $B_s \cup B_{s+1} \cup B_{s+2}$, $s \in \mathbf{Z}$

(v) $\cup_{i=0}^{15} B_{s+i}$

(vi) $\cup_{i \in \mathbf{Z}} B_{s+i}$

38. Let $D_n = [0, 1/n]$, $S_n = (0, 1/n]$ and $T_n = [0, 1/n)$ where $n \in \mathbf{N}$, the positive integers. Find: (i) $\bigcap \{D_n : n \in \mathbf{N}\}$, (ii) $\bigcap \{S_n : n \in \mathbf{N}\}$, (iii) $\bigcap \{T_n : n \in \mathbf{N}\}$.

39. Prove DeMorgan's Laws: (i) $(\cup_i A_i)^c = \cap_i A_i^c$, (ii) $(\cap_i A_i)^c = \cup_i A_i^c$.

40. Let $\mathcal{A} = \{A_i : i \in I\}$ be an indexed class of sets and let $J \subset K \subset I$. Prove:

(i) $\bigcup\{A_i : i \in J\} \subset \bigcup\{A_i : i \in K\}$, (ii) $\bigcap\{A_i : i \in J\} \supset \bigcap\{A_i : i \in K\}$

ASSOCIATED SET FUNCTIONS

41. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$. Find: (i) $f[\{-1, 0, 1\}]$, (ii) $f^{-1}[\{10, 17\}]$, (iii) $f[(-2, 2)]$, (iv) $f^{-1}[(5, 10)]$, (v) $f[\mathbb{R}]$, (vi) $f^{-1}[\mathbb{R}]$.

42. Prove: A function $f : X \to Y$ is one-one if and only if $f[A \cap B] = f[A] \cap f[B]$, for all subsets A and B of X.

43. Prove: Let $f : X \to Y$. Then, for any subsets A and B of X,

(*a*) $f[A \cap B] \subset f[A] \cap f[B]$, (*b*) $A \subset B$ implies $f[A] \subset f[B]$

44. Prove: Let $f : X \to Y$. Then, for any subsets A and B of Y,

(*a*) $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$, (*b*) $A \subset B$ implies $f^{-1}[A] \subset f^{-1}[B]$

45. Prove Theorem 2.8: Let $f : X \to Y$ and let $A \subset X$ and $B \subset Y$. Then

(i) $A \subset f^{-1} \circ f[A]$, (ii) $B \supset f \circ f^{-1}[B]$

46. Prove: Let $f : X \to Y$ be onto. Then the associated set function $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ is also onto.

47. Prove: A function $f : X \to Y$ is both one-one and onto if and only if $f[A^c] = (f[A])^c$ for every subset A of X.

48. Prove: A function $f : X \to Y$ is one-one if and only if $A = f^{-1} \circ f[A]$ for every subset A of X.

ALGEBRA OF REAL-VALUED FUNCTIONS

49. Let $X = \{a, b, c\}$ and let f and g be the following real valued functions on X:

$$f = \{\langle a, 2\rangle, \langle b, -3\rangle, \langle c, -1\rangle\}, \qquad g = \{\langle a, -2\rangle, \langle b, 0\rangle, \langle c, 1\rangle\}$$

Find (i) $3f$, (ii) $2f - 5g$, (iii) fg, (iv) $|f|$, (v) f^3, (vi) $|3f - fg|$.

50. Let A be any subset of a universal set U. Then the real-valued function $\chi_A : U \to \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called the *characteristic function* of A. Prove:

(i) $\chi_{A \cap B} = \chi_A \chi_B$, (ii) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$, (iii) $\chi_{A \setminus B} = \chi_A - \chi_{A \cap B}$.

51. Prove: $\mathcal{F}(X, \mathbb{R})$ satisfies the axiom $[V_2]$ of Theorem 2.9; i.e. if $f \in \mathcal{F}(X, \mathbb{R})$ and $k, k' \in \mathbb{R}$, then (i) $k \cdot (k' \cdot f) = (kk') \cdot f$, (ii) $1 \cdot f = f$

52. For each $k \in \mathbb{R}$, let $\hat{k} \in \mathcal{F}(X, \mathbb{R})$ denote the constant function $\hat{k}(x) = k$ for all $x \in X$.

(i) Show that the collection $\mathcal{C}$ of constant functions, i.e. $\mathcal{C} = \{\hat{k} : k \in \mathbb{R}\}$, is a linear subspace of $\mathcal{F}(X, \mathbb{R})$.

(ii) Let $\alpha : \mathcal{C} \to \mathbb{R}$ be defined by $\alpha(\hat{k}) = k$. Show that α is both one-one and onto and that, for any k, $k' \in \mathbb{R}$,

$$\alpha(\hat{k} + \hat{k}') = \alpha(\hat{k}) + \alpha(\hat{k}')$$

28. A real number x is called *transcendental* if x is not algebraic, i.e. if x is not a solution to a polynomial equation
$$p(x) = a_0 + a_1x + \cdots + a_mx^m = 0$$
with integral coefficients (see Problem 5). For example, π and e are transcendental numbers.
(i) Prove that the set T of transcendental numbers is non-denumerable.
(ii) Prove that T has the power of the continuum, i.e. has cardinality $\mathbf{c}$.

29. An operation of multiplication is defined for cardinal numbers as follows:
$$\#(A)\,\#(B) = \#(A \times B)$$
(i) Show that the operation is well-defined, i.e.,
$$\#(A) = \#(A') \text{ and } \#(B) = \#(B') \quad \text{implies} \quad \#(A)\,\#(B) = \#(A')\,\#(B')$$
or, equivalently, $\quad A \sim A'$ and $B \sim B'$ implies $(A \times B) \sim (A' \times B')$
(ii) Prove: (*a*) $\aleph_0\,\aleph_0 = \aleph_0$, (*b*) $\aleph_0\,\mathbf{c} = \mathbf{c}$, (*c*) $\mathbf{c}\,\mathbf{c} = \mathbf{c}$.

30. An operation of addition is defined for cardinal numbers as follows:
$$\#(A) + \#(B) = \#(A \times \{1\} \cup B \times \{2\})$$
(i) Show that if $A \cap B = \emptyset$, then $\#(A) + \#(B) = \#(A \cup B)$.
(ii) Show that the operation is well-defined, i.e.,
$$\#(A) = \#(A') \text{ and } \#(B) = \#(B') \quad \text{implies} \quad \#(A) + \#(B) = \#(A') + \#(B')$$

31. An operation of powers is defined for cardinal numbers as follows:
$$\#(A)^{\#(B)} = \#(\{f : f: B \to A\})$$
(i) Show that if $\#(A) = m$ and $\#(B) = n$ are finite cardinals, then
$$\#(A)^{\#(B)} = m^n$$
i.e. the operation of powers for cardinals corresponds, in the case of finite cardinals, to the usual operation of powers of positive integers.
(ii) Show that the operation is well-defined, i.e.,
$$\#(A) = \#(A') \text{ and } \#(B) = \#(B') \quad \text{implies} \quad \#(A)^{\#(B)} = \#(A')^{\#(B')}$$
(iii) Prove: For any set A, $\#(\mathcal{P}(A)) = 2^{\#(A)}$.

32. Let $\sim$ be the equivalence relation in $\mathbf{R}$ defined by $x \sim y$ iff $x - y$ is rational. Determine the cardinality of the quotient set $\mathbf{R}/\sim$.

33. Prove: The cardinal number of the class of all functions from $[0,1]$ into $\mathbf{R}$ is $2^{\mathbf{c}}$.

34. Prove that the following two statements of the Schroeder-Bernstein Theorem 3.8 are equivalent:
(i) If $A \precsim B$ and $B \precsim A$, then $A \sim B$.
(ii) If $X \supset Y \supset X_1$ and $X \sim X_1$, then $X \sim Y$.

35. Prove Theorem 3.9: Given any pair of sets A and B, either $A \prec B$, $A \sim B$ or $B \prec A$. (*Hint.* Use Zorn's Lemma.)

ORDERED SETS AND SUBSETS

36. Let $A = (\mathbf{N}, \leqq)$, the positive integers with the natural order; and let $B = (\mathbf{N}, \geqq)$, the positive integers with the inverse order. Furthermore, let $A \times B$ denote the lexicographical ordering of $\mathbf{N} \times \mathbf{N}$ according to the order of A and then B. Insert the correct symbol, $\prec$ or $\succ$, between each pair of elements of $\mathbf{N} \times \mathbf{N}$.

(i) $\langle 3,8\rangle$ ___ $\langle 1,1\rangle$, (ii) $\langle 2,1\rangle$ ___ $\langle 2,8\rangle$, (iii) $\langle 3,3\rangle$ ___ $\langle 3,1\rangle$, (iv) $\langle 4,9\rangle$ ___ $\langle 7,15\rangle$.