MA 1124 Annual Exam 2015

1. (a) Construct the True/False table for $(P \wedge \sim Q) \rightarrow (Q \vee P)$

   (b) Write down the negation of the following statements

      i. $\forall \epsilon \exists N(x,\epsilon)$ such that $N(x,\epsilon) \cap A \neq \Phi$

      ii. If today is monday it is raining.

   (c) State DeMorgan's Laws for logical statements and for sets.

   (d) Define equivalence relation. Define sequence $\{x_n\} \sim$ sequence $\{b_n\}$ means $\lim_{n\to\infty}(a_n - b_n) \rightarrow 0$. Is $\sim$ an equivalence relation?

   (e) Define partial order and state Zorn's Lemma. Define well ordered and total or linear order.

2. (a) Define what it means for two sets to have the same cardinal number. Prove that the set of subsets of $\mathbb{N}$ has the same cardinal number as $\Pi_1^\infty\{0,1\}$.

   (b) State the Schroeder-Bernstein Theorem and Cantor's Theorem. Prove the Schroeder-Bernstein Theorem.

3. (a) Prove that the greatest lower bound of a bounded set is a limit point of that set. Prove that the greatest lower bound of an open bounded set is not a member of the set. Prove that $\mathbb{R}$ and Φ are the only subsets of $\mathbb{R}$ that are open and closed.

   (b) Prove that any subset of $\mathbb{R}$ has the property that every open covering has a finite subcover if and only if it is closed and bounded.

   (c) Prove that the intersection of any collection of closed sets is closed.

4. (a) Prove that f : $\mathbb{R} \rightarrow \mathbb{R}$ is continuous if and only if the inverse image of every open set is open.

   (b) Prove that the continuous image of any compact set is compact. Hence deduce the extreme value theorem.

   (c) Define what it means for a subset of $\mathbb{R}$ to be connected. Given that the only subsets of $\mathbb{R}$ that are both open and closed are $\mathbb{R}$ and the empty set, prove that $\mathbb{R}$ is connected. Prove that the continous image of any connected set is connected. Hence or otherwise prove that every interval is connected.

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Q 1 (a)

P	Q	$\sim Q$	$P \wedge \sim Q$	$Q \vee P$	$(P \wedge \sim Q) \to (Q \vee P)$
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	F	T

(b) (i) $\exists \epsilon \; \forall N(x, \epsilon) \quad N(x, \epsilon) \cap A = \phi$.

(ii) Today is monday and it is not Raining

(c) $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$.

$\sim(P \vee Q) \equiv \sim P \wedge \sim Q$.

$(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

(d) An equivalence relations is a binary relation $a \sim b$ such that $a \sim a \;\; \forall a$

$a \sim b \to b \sim a \;\; \forall a, b$.

$a \sim b \;\&\; b \sim c \to a \sim c \;\; \forall a, b, c$.

$\{a_n\} \sim \{b_n\}$ means $\lim_{n\to\infty} (a_n - b_n) \to 0$.

$\{a_n\} \sim \{a_n\}$ since $\lim_{n\to\infty} (a_n - a_n) = 0$.

$\{a_n\} \sim \{b_n\} \Rightarrow \lim_{n\to\infty} (a_n - b_n) = 0$

$\Rightarrow \lim_{n\to\infty} -(a_n - b_n) = -0$
$= 0$.

$\therefore \{b_n\} \sim \{a_n\}$

$\{a_n\} \sim \{b_n\} \Rightarrow \lim (a_n - b_n) = 0$

$\{b_n\} \sim \{c_n\} \Rightarrow \lim (b_n - c_n) = 0$

$\lim (a_n - \cancel{b_n}) = \lim a_n - c_n + c_n - b_n$

$= \lim a_n - c_n + \lim c_n - b_n$

$= 0 + 0 = 0$

Hence we have an equivalence relation

(e) A partial order on X is a binary relation $\sim$ such that $a \sim a \ \forall a \in X$.

$a \sim b + b \sim a \Rightarrow b = a$

$a \sim b + b \sim c \Rightarrow a \sim c$.

Zorns Lemma Let $\leq$ be a partial order on a set X. If every totally ordered subset of X has an upper bound then X contains a maximal element.

$\leq$ is a total order on X is $\leq$ is a partial order such that $\forall a, b \in X$ $a \leq b$ or $b \leq a$.

$X, \leq$ is well ordered if $\leq$ is a total ordered such that every non-empty subset of X has a least element.

2. (a) Two sets have the same cardinal number means there is a bijective mapping between them.

We want a bijective mapping from the subsets of $\mathbb{N}$ to sequences of 0's and 1's $= \prod_{1}^{\infty} \{0,1\}$.

If $A \subset \mathbb{N}$, define

$$f(A) = \{n_i\} \quad \text{where } n_i = 1 \text{ if } i \in A$$
$$= 0 \text{ if } i \notin A.$$

We need to show f is 1-1 and onto.

Let $f(A) = f(B)$ i.e. $\{n_i\} = \{m_i\}$

Let $j \in A \Rightarrow n_j = 1 \Rightarrow m_j = 1 \Rightarrow j \in B$.

Sim $j \notin A \Rightarrow n_j = 0 \Rightarrow m_j = 0 \Rightarrow j \notin B$

$\therefore A = B$. So 1-1

2 (a) continued.

Let $\{n_i\} \in \prod_{i=1}^{\infty} \{0,1\}$.

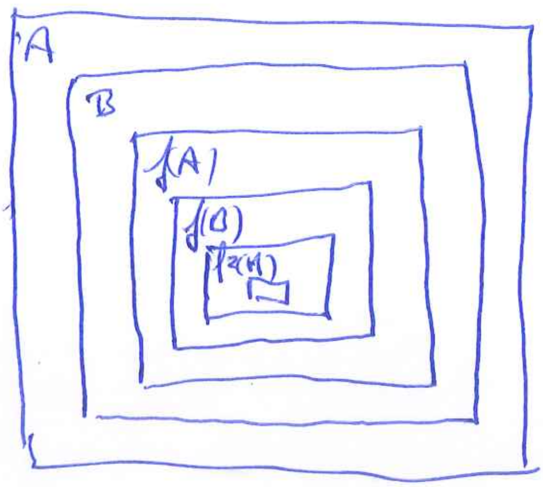
Let $A = \{i : n_i = 1\}$ then $f(A) = \{n_i\}$

so f is onto.

(b) Schroeder-Berstein Thm. Given sets A and B, if $\exists f : A \to B$ 1-1 + $\exists g : B \to A$ 1-1 then $\exists h : A \to B$, 1-1 + onto.

Cantor's Thm. Let A be any set, then the cardinality of the set of subsets of A is of greater cardinality than A.

Proof of Schroeder-Berstein: We can reformulate this at $A \subset g(B) = B$ and $f : A \to B$. Then the picture is



etc

2 (b) continued

We have

$$A = A \setminus B \cup B \setminus f(A) \cup f(A) \setminus f(B) \cup f(B) \setminus f^2(A) \cup \ldots$$

and $\quad h = id \quad h = f \qquad \cup \bigcap f^n(A) \quad h = id \quad h = f$.

$$B = B \setminus f(A) \cup f(A) \setminus f(B) \cup f(B) \setminus f^2(A) \cup f^2(A) \setminus f^2(B) \ldots$$

$$\cup \bigcap f^n(A)$$

define $h : A \to B$ as above

and $h : \bigcap f^n(A) \to \bigcap f^n(A) = id$.

Then h is clearly <u>onto</u> <u>and</u> since all the pieces are disjoint and h is 1-1 on each piece then h is 1-1 on A.

3 (a) Let $A \subset \mathbb{R}$ be bounded and $c = g.l.b.(A)$.

then either $c \in A$ or c is a limit point of A, i.e. $\exists a_n \in A$ $a_n \neq c$ s.t. $a_n \to c$.

If $c \notin A$, then $c + 1$ is not a lower bound for A so $\exists a_1 \in A$ s.t. $c < a_1 < c + 1$. Then $c + 1/2$ and a_1 are not lower bounds for A so $\exists\ a_2 \in A$ s.t.

$$c < a_2 \leq \min(a_1, c + 1/2).$$

Continue set $c \leq a_n \leq c + \frac{1}{n}$, $a_n \in A$

so $a_n \to c$.

If A is open, and $c \in A$, then $\exists\ \epsilon$

s.t. $c - \epsilon \in A$ also; so c is not

a lower bound. Contradiction so $c \notin A$.

Let $A \subset \mathbb{R}$ be open and closed.

$A \neq \mathbb{R}$ or $\phi \Rightarrow \exists\ a \notin A$.

and one of $(-\infty, a)$ or $(a, +\infty)$ must

intersect A. In the second case

let $c = \text{g.l.b.}(a, +\infty) \cap A = \text{g.l.b}\,[a, +\infty) \cap A$.

then $c \notin (a, +\infty) \cap A$ since open

But $c \in [a, +\infty) \cap A$ since closed

$\Rightarrow \Leftarrow$.

(b) Let $A \subset \mathbb{R}$ be closed + bounded

Case 1 $A = [a, b]$.

Suppose $A \subset \bigcup O_\alpha$ O_α open and

there is no finite subcover.

Let $I_1 = [a, b]$.

3 (b) Then consider $[a, \frac{a+b}{2}]$, and $[\frac{a+b}{2}, b]$
One of these cannot have a finite subcover since if they both did so would $[a, b]$. Let I_2 be one without a finite subcover.
Keep splitting. Get

$$I_1 \supset I_2 \supset I_3 \cdots$$

s.t. $$|I_n| = \frac{|[a, b]|}{2^{n-1}} = \frac{b-a}{2^{n-1}}$$

and each I_n cannot be covered by a finite number of the O_α's.

By the Nested Intervals Thm,
$\cap I_n \neq \phi$. In fact since $|I_n| \to 0$ $\cap I_n = \{x\}$ some x.

Now $x \in A \to \exists\, O_{\alpha_0}$ s.t. $x \in O_{\alpha_0}$
O_{α_0} open $\to \exists\, \epsilon$ s.t. $(x-\epsilon, x+\epsilon) \subset O_{\alpha_0}$. But if $|I_n| < \epsilon$, and $x \in I_n\ \forall n$, then $I_n \subset O_{\alpha_0}$

3 (b)
so I_n has a finite subcover $\Rightarrow \Leftarrow$.
Hence $[0, b]$ has a finite subcover.

<u>Case 2</u> A closed bounded

A bounded $\rightarrow \exists\, [a, b]$ s.t $A \subset [a, b]$.

A closed $\rightarrow A^c$ open.

Then $A^c \cup \bigcup O_\alpha \supset [a, b]$, so

$[a, b]$ has a finite subcover

$A^c \cup O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$.

So $A \subset A^c \cup O_{\alpha_1} \cdots \cup O_{\alpha_n}$

But $A \cap A^c = \emptyset$

so $A \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$
as required.

Conversely if A is not bounded
- above say. Then

$(-\infty, n)$ cover A, but
no finite subcover works.

and if A is not closed, $\exists\, x$ an
accumulation point of A, $x \notin A$.

3 (A) Then $O_n = (-\infty, x - 1/n) \cup (x + 1/n, +\infty)$
cover A but since A contains points arbitrarily ~~closer~~ close to x, no finite subcover exists.

(C) Let O_α be open
Let $A = \bigcup O_\alpha$, let $x \in A$

$\Rightarrow x \in O_{\alpha_0}$ some α_0

$\Rightarrow \exists N(x, \epsilon) \subset O_{\alpha_0}$ some ϵ, since O_{α_0} open

$\Rightarrow \exists N(x, \epsilon) \subset A$

$\Rightarrow A$ is open

Let G_α be closed,

$(\bigcap G_\alpha)^c = \bigcup G_\alpha^c$

$= \bigcup$ open $=$ open

$\Rightarrow \bigcap G_\alpha = $ closed.

4. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous
Let $O \subset \mathbb{R}$ be open
Claim $f^{-1}(O)$ is open
Let $x \in f^{-1}(O)$

4 (a)

$\Rightarrow$ $f(x) \in O \Rightarrow \exists\, \epsilon$ s.t $(x-\epsilon, x+\epsilon) \subset O$ open

$\Rightarrow$ by cont. $\exists\, \delta$. s.t $0 < |y-x| < \delta$

$\Rightarrow |f(y) - f(x)| < \epsilon$

$\Rightarrow$ $f(y) \in O$

$\Rightarrow$ $N(x, \delta) \subset f^{-1}(O)$.

Conversly given $\epsilon > 0$ and any x

$O = N(f(x), \epsilon)$ is open

then $f^{-1}(O)$ is open and $x \in f^{-1}(O)$

so $\exists\, \delta$ s.t $N(x, \delta) \subset f^{-1}(O)$

$\Rightarrow 0 < |y - x| < \delta \Rightarrow f(y) \in N(f(x), \epsilon)$

$\Rightarrow |f(y) - f(x)| < \epsilon$

So f is cont.

(b) Let A be compact. Claim f cont.

$\Rightarrow f(A)$ compact.

Let $f(A) \subset \bigcup O_\alpha$ open cover

then $A \subset f^{-1}(\bigcup O_\alpha) = \bigcup f^{-1}(O_\alpha)$

= open cover

$\Rightarrow \exists\, f^{-1}(O_{\alpha_1}) \cup \quad \cup f^{-1}(O_{\alpha_n}) \supset A$

$\Rightarrow$ $O_{\alpha_1} \cup \quad \cup O_{\alpha_n} \supset f(A)$

Hence $f(A)$ is compact

4 (b) Let f be cont. on $[a,b]$.

then by the above $[a,b]$ is compact

$\Rightarrow$ $f([a,b])$ is compact

$\Rightarrow$ $f([a,b])$ is closed and bounded

bounded $\Rightarrow$ $f([a,b])$ has a g.l.b + a l.u.b.

closed $\Rightarrow$ these belong to $f([a,b])$

So the max and min. are assumed.

ie $\exists$ x_0 and x_1 in $[a,b]$

$f(x_0) = \max$ + $f(x_1) = \min$

(c) $A \subset \mathbb{R}$ is disconnected means $\exists$ O_1 + O_2 open s.t.

$A \subset O_1 \cup O_2$

$A \cap O_i \neq \emptyset$ $\quad i = 1, 2$

$O_1 \cap O_2 = \emptyset$.

A is connected means A is not disconnected.

4 (c) If $\mathbb{R}$ is disconnected

$$\mathbb{R} = O_1 \cup O_2 \qquad O_i \text{ open}$$

and $O_1 \cap O_2 = \emptyset$

$$\Rightarrow O_1 = O_2^c$$

$\Rightarrow O_1 + O_2$ are both open + closed

$\Rightarrow$ one $= \mathbb{R}$ + other $= \emptyset$

Hence $\mathbb{R} \cap O_1 = \emptyset$ or $\mathbb{R} = O_2 = \emptyset$.

Contradiction, so $\mathbb{R}$ is connected.

A connected, claim $f(A)$ is connected

If $f(A)$ is disconnected,

$$f(A) \subset O_1 \cup O_2$$

then $A \subset f^{-1}(O_1) \cup f^{-1}(O_2)$ both open

$$O_1 \cap O_2 = \emptyset$$

$$\Rightarrow f^{-1}(O_1) \cap f^{-1}(O_2) = \emptyset.$$

and $O_i \cap f(A) \neq \emptyset$

$$\Rightarrow f^{-1}(O_i) \cap A \neq \emptyset$$

$\Rightarrow A$ is disconnected $\Rightarrow\Leftarrow$

So $f(A)$ is connected

4 (c) $\tan^{-1} x : \mathbb{R} \to (-\pi/2, \pi/2)$ conts

So $(-\pi/2, \pi/2)$ is connected.

Can then show $[-\pi/2, \pi/2)$ is connected

then $[-\pi/2, \pi/2]$ is connected

and from these any interval

is a continuous image by a

linear map.