MA1124 Exam. May 2013

1. (a) Use True/False tables to prove the following

    i. $\sim Q \rightarrow \sim P \equiv P \rightarrow Q$ What is this called?

    ii. $\sim (P \wedge Q) \equiv \sim P \vee \sim Q$ What is this called?

   (b) Write down the negation of the following statement in logical terms

   $\lim_{x \to a} f(x) = L$

   (c) Is the following a valid argument? If John solved the problem correctly then he obtained the answer 3. He got the answer 3, so he solved the problem correctly.

2. (a) Define what it means for two sets to have the same cardinal number. Define what it means for one cardinal number to be greater than or equal to another. State the Schroeder Bernstein Theorem.

   (b) Define what it means for a set to be countable. Show that the set $\Pi_{i=1,\infty}\{1,0\}$ is equivalent to the set of subsets of Z^+, and explain why this shows it is uncountable.

   (c) Define partially ordered set and totally ordered set. State Zorn's Lemma, and name three other equivalent statements.

3. (a) State the least upper bound axiom for the real numbers. State and prove the Nested Intervals Theorem.

   (b) Define open set in $\mathbb{R}$. State and prove what you know about unions and intersections of open sets.

   (c) Define $\overline{A}$, the closure of a A a subset of $\mathbb{R}$, and A°, the interior of A. Prove $(A^\circ)^c = \overline{(A^c)}$.

   (d) Define connected subset of $\mathbb{R}$. Sketch the proof of the Intermediate Value Theorem.

4. (a) Prove the following for A a subset of $\mathbb{R}$, if A is closed and bounded then every sequence has a convergent subsequence.

   (b) Prove that every Cauchy sequence in $\mathbb{R}$ converges.

   (c) Define what it means for a to be a boundary point of A, a subset of $\mathbb{R}^2$.

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Solutions

1 (a) (i)

P	Q	$P \to Q$	$\sim Q$	$\sim P$	$\sim Q \to \sim P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The columns $P \to Q$ and $\sim Q \to \sim P$ are equal so these are equivalent statements.

$\sim Q \to \sim P$ is the contrapositive of $P \to Q$.

(ii)

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\ast \quad \equiv \quad \ast$

$\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ is one of De Morgan's Laws.

(b) $\lim_{x \to a} f(x) = L$ means $\forall \epsilon > 0, \exists \delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$

So the negation is

$$\exists \epsilon > 0, \forall \delta > 0 \quad \underbrace{0 < |x-a| < \delta \not\Rightarrow |f(x) - L| < \epsilon}_{\equiv}$$

$$\exists x_0 \quad 0 < |x_0 - a| < \delta, \text{ but } |f(x_0) - L| \geq \epsilon$$

(c) The argument is invalid. You are given "If John solved the problem correctly then he obtained the answer 3". This does not tell you that if he obtained the answer three then he solved the problem correctly. These are converse statements.

2. (a) Two sets have the same cardinal number means there exists a bijective mapping from one to the other.

If $\# X = n$ and $\# Y = m$, then $n \leq m$ means there exists a 1-1 mapping from X to Y.

The Schroder-Bernstein theorem states that if $n \leq m$ and $m \leq n$ then $n = m$.

(b) A set X is countable if its cardinality is $\leq$ the cardinality of $\mathbb{Z}^+$

We want a function that is bijective from $\prod_{i=1,\infty} \{1,0\}$ to the set of subsets of $\mathbb{Z}^+$.

Let $Y = \{Y_i\}_{i=1}^{\infty} \in \prod_{i=1,\infty} \{1,0\}$

Define $f(Y) = \{i \in \mathbb{Z}^+ : \cancel{y(i)}\ Y_i = 1\}$.

Example

$Y = \{0, 1, 0, 0, 1, 1, 0, 0, \ldots\}$

then $f(Y) = \{2, 5, 6\} \subset \mathbb{Z}^+$

Claim $f(Y)$ is 1-1.

Let $f(Y) = f(\bar{Y})$.

then $Y_i = 1 \iff \bar{Y}_i = 1$

Hence $\{Y_i\} = \{\bar{Y}_i\}$.

Claim f is onto

Let $A \subset \mathbb{Z}^+$.

Define $Y = \{Y_i\}$ where $Y_i = 1 \iff i \in A$

then clearly $f(Y) = A$.

By Cantor's Theorem $\# \mathcal{P}(A) > \# A$ any set A, so $\# \prod_{i=1,\infty} \{1,0\} > \# \mathbb{Z}^+$

Hence it is uncountable.

(c) A set X has a partial order $\leq$ if it $\leq$ is a relation on X such that $x \leq X \quad \forall x \in X$.

$$x \leq y \text{ and } y \leq x \Rightarrow y = x.$$

$$x \leq y \text{ and } y \leq z \Rightarrow x \leq z.$$

$(X, \leq)$ is totally ordered if $\forall x, y \in X$, $x \leq y$ or $y \leq x$.

Zorn's Lemma: If every totally ordered subset of $(X, \leq)$ has an upper bound, then $(X, \leq)$ has a maximal element.

Equivalent statements are 1. The Axiom of Choice, 2. Transfinite Induction and 3 that every set can be well ordered.

3. (a) The least upper bound axiom for $\mathbb{R}$ states that every subset which has an upper bound has a least upper bound.

The Nested Intervals Theorem. If I_n are a sequence of closed bounded intervals in $\mathbb{R}$ with $I_{n+1} \subset I_n$. then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. I_n is a closed bounded interval, so

$$I_n = [a_n, b_n] \text{ some } a_n, b_n \in \mathbb{R}.$$

Now $I_{n+1} \subset I_n \Rightarrow a_{n+1} \geq a_n$

$b_{n+1} \leq b_n$.

Claim $a_n \leq b_m \quad \forall n, m$.

1. $n \leq m \quad a_n \leq a_m \leq b_m$.

2. $m \leq n \quad a_n \leq b_n \leq b_m$

Hence $\{a_n\}$ is bounded above, $\{b_n\}$ is bounded below.

Now $a_n \leq l.u.b. \{a_n\} \leq b_m \ \forall m$.

~~g.l.b.~~

$\therefore$ $l.u.b. \{a_n\} \leq g.l.b. \{b_m\}$
$\quad\quad \shortparallel \quad\quad\quad\quad \shortparallel$
$\quad\quad a$ say $\quad\quad b$ say

Hence $[a, b] \subset I_n \quad \forall n$

$\therefore \bigcap I_n \neq \emptyset$

<u>More</u> it contains $[a, b]$.

(b) O is open in $\mathbb{R}$ if every point is an interior point i.e. $\forall x \in O$, $\exists \epsilon > 0$ s.t. $N(x, \epsilon) \subset O$.

Claim $\bigcup_\alpha O_\alpha$ is open if O_α are open

Proof: Let $x \in \cup O_\alpha$, $\Rightarrow \exists \alpha_0$ s.t. $x \in O_{\alpha_0}$

$\Rightarrow \exists \epsilon_0$, $N(x, \epsilon_0) \subseteq O_{\alpha_0}$

$\Rightarrow N(x, \epsilon_0) \subset \cup O_{\alpha_0}$

Hence $\cup O_\alpha$ is open.

Let $O_{\alpha_1}, \dots, O_{\alpha_n}$ be open, claim $\cap O_{\alpha_i}$ is open

<u>Proof</u>; Let $x \in \cap O_{\alpha_i} \Rightarrow x \in O_{\alpha_i}$ all $i = 1, n$

$\Rightarrow \exists \epsilon_i$, s.t. $N(x, \epsilon_i) \subset O_{\alpha_i}$

Let $\epsilon = \min_{i=1,n} \{\epsilon_i\}$

$\Rightarrow N(x, \epsilon) \subset \cap O_{\alpha_i}$

Hence $\bigcap_{i=1}^{n} O_{\alpha_i}$ is open

But Let $O_{\alpha_n} = \left(-\frac{1}{n}, \frac{1}{n}\right)$

then $\cap O_{\alpha_n} = \{0\}$ – not open

So only <u>finite</u> intersections of open sets are open.

(c) $\bar{A} = A \cup$ (accumulation points of A)

$= \{x : N(x, \epsilon) \cap A \neq \emptyset, \forall \epsilon > 0\}$

A° = set of interior points of A

$= \{x \in A : \exists \epsilon$ s.t. $N(x, \epsilon) \subset A\}$

Claim $(A^{o})^{c} = \overline{(A^{c})}$

Proof: Let $x \in (A^{o})^{c} \Leftrightarrow x \notin A^{o}$

$\Leftrightarrow \forall \epsilon > 0, N(x, \epsilon) \cap A^{c} \neq \emptyset$

$\Leftrightarrow x \in \overline{A^{c}}$.

(d) $A \subset \mathbb{R}$ is disconnected means $\exists O_1, O_2$ open in $\mathbb{R}$ s.t. $A \subseteq O_1 \cup O_2$

$A \cap O_i \neq \emptyset, i = 1, 2$

$O_1 \cap O_2 = \emptyset$.

A is connected if A is not disconnected

The Intermediate Value Theorem says that if $f: \mathbb{R} \to \mathbb{R}$ is continuous, and $f(a) = x_1$ and $f(b) = x_2$ $a < b$ then $\forall x_1 \leq x_3 \leq x_2$ $\exists a \leq c \leq b$ such that $f(c) = x_3$.

Sketch of Proof:

$[a, b]$ is an interval, hence can be shown to be connected. The continuous image of any connected set can be shown to be connected. Hence $f([a, b])$ is connected. But the only connected subsets of $\mathbb{R}$ are intervals. So x_3 is in $f([a, b])$

4. (a) Let A be closed and bounded in $\mathbb{R}$. Then $A \subset [a,b]$ some a, b. Let $[a,b] = I_1$. Let $\{x_n\} \in A$. Claim $\exists\, x_{n_i} \to x \in A$.

Consider $[a, \frac{a+b}{2}]$ + $[\frac{a+b}{2}, b]$.

One (at least) of these contains an infinite number of elements of $\{x_n\}$. Call it I_2. Split I_2 in half and set I_3 with an infinite number of elements of $\{x_n\}$. Continue.

We obtain $I_1 \supset I_2 \supset \quad \supset I_n \supset I_{n+1} \cdots$

I_n closed bounded intervals. By the Nested Intervals Theorem $\bigcap I_n \neq \emptyset$. Let $x \in \bigcap I_n$.

$\forall\, I_i$, choose $x_{n_i} \in I_i$ such that $n_i > n_{i-1}$. This is always possible since we have an infinite number to choose from.

Since $|x_{n_i} - x| < \frac{b-a}{2^i}$, we see that $x_{n_i} \to x$ as desired. Note $x \in A$ as A is closed.

(b) Every Cauchy sequence in $\mathbb{R}$ converges

Proof We can show that every Cauchy sequence is bounded. Hence $\{a_n\}$ is closed and bounded. Hence by (a)

$\exists\ a_{n_i} \to x$. Claim $a_n \to x$.

Given $\epsilon > 0$, $\exists\ N_1$ s.t $n, m \geq N$

$\Rightarrow |a_n - a_m| < \epsilon/_2$.

since $\{a_n\}$ is a Cauchy sequence

and since $a_{n_i} \to x$ $\exists\ N_2$ such that

$n_i \geq N_2 \Rightarrow |a_{n_i} - x| < \epsilon/_2$.

Hence if $n \geq N_3 = \max N_1, N_2$ then.

$|a_n - x| \leq |a_n - a_{n_i}| + |a_{n_i} - x|$

$\leq \epsilon/_2 + \epsilon/_2$

any $n_i \geq N_3$.

Hence $a_n \to x$.

(C) a is a boundary point of $A \subset \mathbb{R}^{\mathbb{Z}}$

if $~~a~~$ $\in \overline{A}$ and $a \in \overline{A^c}$