1. (a) Use True/False tables to prove the following

   5 i. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.

   5 ii. $(P \to Q) \wedge (R \to Q) \equiv (P \vee R \to Q)$.

   (b) Write down the negation of the following statements

   5 i. If f is continuous and O is open then $f^{-1}(O)$ is open.

   5 ii. $\forall x \exists y$ such that $P(x, y) \to Q(x, y)$.

   5 (c) Define what it means for two sets to have the same cardinal number.

2. (a) Define what it means for a set to be countable. Prove that if X and Y are countable sets then the set $X \times Y$ is countable. Is the set $\prod_1^\infty \{0, 1\}$ countable? Prove or disprove. 2 2 6

   (b) Define partially ordered set and totally ordered set. State Zorn's Lemma. 2 2 2

   (c) State the Schroeder-Bernstein Theorem and Cantor's Theorem. Prove one of them. 2 2 5

3. (a) State the least upper bound axiom for the real numbers. Use it to prove the Nested Intervals Theorem. 2 4

   (b) Define open subset, closed subset and compact subset of $\mathbb{R}$. Prove that a subset is compact if and only if it is closed and bounded. 1 1 2 5 4

   (c) Define A^o, and $\overline{A}$, for A a subset of $\mathbb{R}$, and prove $(A^o)^c = \overline{A^c}$. 2 2 2

4. (a) Prove that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. 8

   (b) Prove that every Cauchy sequence in $\mathbb{R}$ converges in $\mathbb{R}$. 3 4 3 9

   (c) Define uniformly continuous function on a subset of $\mathbb{R}$. And prove that any continuous function on a compact subset is uniformly continuous there. 2 6

MA1124 May 2014
Solutions

1(a)(i)

P	Q	R	$P\vee Q$	$P\vee R$	$(P\vee Q)\wedge(P\vee R)$	$Q\wedge R$	$P\vee(Q\wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F
					*		*

So $P\vee(Q\wedge R) \equiv (P\vee Q)\wedge(P\vee R)$

(ii)

P	Q	R	$P\to Q$	$R\to Q$	$(P\to Q)\wedge(R\to Q)$	$P\vee R$	$(P\vee R)\to Q$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	F	T	F
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	F	F	T	F
F	F	F	T	T	T	F	T
					*		*

So $(P\to Q)\wedge(R\to Q) \equiv (P\vee R)\to Q$

(b) (i) If f is cont. and O is open then $f^{-1}(O)$ is open.

The negation is

For some f cont. and O open, $f^{-1}(O)$ is not open.

(ii) $(\forall x \exists y \; P(x,y) \to Q(x,y))'$

$= \exists x \forall y \; P(x,y) \not\to Q(x,y)$

(c) Two sets have the same cardinal number means there exists a bijective (1-1 & onto) mapping between them.

2.(a) A set X is countable if it has the same cardinal number as a subset of $\mathbb{Z}^+$

If X and Y are countable then they can be written $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ and $Y = \{y_1, y_2, \ldots y_n \ldots\}$ where X and Y might be finite.

Then $X \times Y = \{(x_i, y_j)\}$ and we may write the elements in an array

$$\begin{array}{llll} (x_1,y_1), & (x_1,y_2), & \cdots & (x_1,y_n)\cdots \\ (x_2,y_1) & (x_2,y_2), & \cdots & (x_2,y_n)\cdots \\ \vdots & & & \\ (x_n,y_1), & (x_n,y_2) & \cdots & (x_n,y_n)\cdots \end{array}$$

We can then list the pairs as indicated.

The set $\prod_1^\infty \{0,1\}$ is uncountable. We can prove this in various ways.

1 We can show that $\prod_1^\infty \{0,1\}$ is in 1-1 correspondence with the subsets of $\mathbb{Z}^+$ and then use Cantor's Theorem — see notes OR.

2. If $\prod_1^\infty \{0,1\}$ can be listed

$x_1 = x_{11}, x_{12}, x_{13}, \ldots, x_{1n}, \ldots$

$x_2 = x_{21}, x_{22}, x_{23}, \ldots, x_{2n}, \ldots$

$\vdots$

$x_n = x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots$

$\vdots$

where each $x_{ij} = 0$ or 1.

Let $y = \overline{x_{11}}, \overline{x_{22}}, \ldots, \overline{x_{nn}}, \ldots$

where $\overline{x_{ii}} = 1$ if $x_{ii} = 0$

$= 0$ if $x_{ii} = 1$.

then $y \in \prod_1^\infty \{0,1\}$, but y is not on our list.

(b) A set X and a relation $\leq$ is a partially ordered set $(X, \leq)$ if

(i) $x \leq x \ \forall x \in X$

(ii) $x \leq y$ and $y \leq x \rightarrow x = y$

(iii) $x \leq y$ and $y \leq z \rightarrow x \leq z$.

A partially ordered set is totally ordered if $\forall x, y \in X$, $x \leq y$ or $y \leq x$.

Zorn's Lemma: If $(X, \leq)$ is a partially ordered set such that each totally ordered subset has an upper bound, then X has a maximal element.

(c) Schroeder-Bernstein. For two sets X and Y, if $\exists f: X \to Y$ injective and $g: Y \to X$ injective, then $\exists h: X \to Y$, bijective.

(d) Cantor's Theorem: For any set X the cardinal number of the power set (set of all subsets) of X is greater than that of X.

For proofs - see text book.

3 (a) Least Upper Bound Axiom for $\mathbb{R}$:
if $A \subset \mathbb{R}$ is bounded then A has a least upper bound.

Nested Intervals Theorem - see book.

(b) $A \subset \mathbb{R}$ is open means every point in A is an interior point

i.e. $\forall x \in A \ \exists \epsilon > 0$ s.t.

$$B_\epsilon(x) \subseteq A.$$

$A \subset \mathbb{R}$ is closed means that if x is a limit point of A then $x \in A$.

$A \subset \mathbb{R}$ is compact means that every open cover of A has a finite subcover.

If A is closed + bounded then A is compact = Heine-Borel
[but slightly more general than in the book].

Case 1 Let $A = [a, b] = I_1$, $A \subset \bigcup O_{\alpha_i}$

assume no finite subcover.

Consider $[a, \frac{a+b}{2}]$ & $[\frac{a+b}{2}, b]$. If

both of these have a finite subcover

then so does A. So one (at

least) does not have a finite

subcover call it I_2. Keep

going. Get a sequence

$I_1 \supset I_2 \quad \supset I_n \supset I_{n+1} \ldots$

If this stops we have a finite

subcover for A.

By Nested Intervals Theorem

$\bigcap I_n \neq \phi$, say $p \in \bigcap I_n$.

then $p \in$ some O_{α_1} - open

$\exists \; B_\epsilon(p) \subset O_{\alpha_1}$.

$p-\epsilon \quad p \quad p+\epsilon$

Now $p \in I_n \; \forall n$, and $|I_n| \to 0$.

but if $|I_m| < \epsilon$ then $I_m \subset B_\epsilon(p)$

and hence $I_m \subset O_{\alpha_1}$, contradiction

no finite subcover.

Case 2. A closed + bounded. Then $\exists$ $[a,b] \supset A$. ~~Let~~ add A^c to $\{O_\alpha\}$ and we have an open cover of $[a,b]$. Apply Case 1 $\exists$ $A^c \cup O_{\alpha_{i_1}} \cup O_{\alpha_{i_2}} \cup \cup O_{\alpha_{i_n}} \supset A$.

But $A^c \cap A = \phi$

so $A \subset O_{\alpha_{i_1}} \cup O_{\alpha_{i_2}} \cup O_{\alpha_{i_n}}$

Conversely if A is not bounded

e.g. not bounded above

then $A \subset \bigcup (-\infty, n)$ and it cannot have a finite subcover

and if A is not closed. Let x be an accumulation point not in A. Then

$$A \subset \bigcup (-\infty, x - 1/n) \cup (x + 1/n, \infty)$$

is an open cover with no finite subcover.

(c) $A^\circ = \{x \in A : x$ is an interior point$\}$

$\bar{A} = \{x : x$ is a limit point of $A\}$

$= A \cup$ acc points of A.

Let $x \in (A^\circ)^c$, then $x \notin A^\circ$

$\iff \forall \epsilon, B_\epsilon(x) \cap A^c$.

$\iff x$ is a limit point of A^c

$\iff x \in \overline{A^c}$,

Hence $(A^\circ)^c = \overline{A^c}$.

4. (a) Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Then $\exists I_1 = [a, b]$ s.t. $\{x_n\} \subset I_1$. Consider $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. One of these (at least) must contain an infinite number of terms of the sequence. Call this I_2. Proceed. We get

$I_1 \supset I_2 \quad \supset I_n \supset I_{n+1} \supset \cdots$

and each I_n contains

an infinite number of the x_n's.

I_n closed nested intervals $\Rightarrow \bigcap I_n \neq \emptyset$.

Let $p \in \bigcap I_n$. Choose any x_{n_1} in I_1.

Choose x_{n_2} with $n_2 > n_1$ in I_2

and having chosen x_{n_i} in I_i,

Choose $x_{n_{i+1}}$ ~~~~ in $I_{n_{i+1}}$ with

$n_{i+1} > n_i$. We can do

this at each stage since

each I_n contains infinitely

many x_n's.

Since $|I_n| \to 0$ we see

that $x_{n_i} \to p$ and we

have our convergent

subsequence.

(iv) Every cauchy sequence in $\mathbb{R}$ is

bounded

PROOF:

Let $\epsilon = 1$, $\exists N$ s.t $n, m \geq N \rightarrow$
$|x_n - x_m| < 1$. In particular
$\forall n \geq N \quad |x_n - x_N| < 1$.

Let $M = \max\{x_1, x_2, \ldots, x_N, x_N + 1\}$

then $x_n \leq M \quad \forall n$.

Let $m = \min\{x_1, x_2, \ldots, x_N, x_N - 1\}$

then $x_n \geq m \quad \forall n$.

So our Cauchy sequence is bounded

Apply part (a) to get a
convergent subsequence $x_{n_k} \rightarrow x$ say.

Next claim all of $x_n \rightarrow x$.

Given $\epsilon > 0$ $\exists N_1$ s.t

$n, m \geq N_1 \rightarrow |x_n - x_m| < \epsilon/2$

this is because $\{x_n\}$ is a
Cauchy sequence.

+ $x_{n_k} \rightarrow x$ so $\exists N_2$ s.t $x_{n_k} \geq N_2$
$\rightarrow |x_{n_k} - x| < \epsilon/2$

Let $N = \max(N_1, N_2)$, choose n_{i_0} with $n_{i_0} > N$ then $\forall n \geq N$

$$|x_n - x| \leq |x_n - x_{n_{i_0}}| + |x_{n_{i_0}} - x|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $x_n \to x$.

(c) f is uniformly continuous on $A \subset \mathbb{R}$ means $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $0 < |x_1 - x_2| < \delta$ $\rightarrow |f(x_1) - f(x_2)| < \epsilon$.

Let A be compact, f continuous on A. $\forall \epsilon$, $\forall x \ \exists \delta_x$ s.t.

$$0 < |x - y| < \delta_x \rightarrow |f(x) - f(y)| < \epsilon/2.$$

Now $\{B_{\frac{1}{2}\delta_x}(x)\}_{x \in A}$ are an open cover for A so we have a finite subcover $B_{\frac{1}{2}\delta_{x_1}}(x_1), \ldots, B_{\frac{1}{2}\delta_{x_n}}(x_n)$.

Let $x_1, x_2 \in A$ with $|x_1 - x_2| < \delta = \min_{i=1,n} \delta_{x_i}$

$x_1 \in B_{\frac{1}{2}\delta_{x_{i_0}}}(x_{i_0})$ some x_{i_0}.

and $|x_1 - x_2| < \frac{1}{2}\delta_{x_{i_0}}$, so

x_2 also is in $B_{\delta_{x_{i_0}}}(x_{i_0})$. Hence

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(x_{i_0})| + |f(x_{i_0}) - f(x_2)|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$