1. $\int \cos^5 x \, dx$ Let $u = \sin x$

$\frac{du}{dx} = \cos x$

$$\int \cos^4 x \,.\, \frac{du}{dx}\, dx = \int (1-\sin^2 x)^2 \frac{du}{dx}\, dx$$

$$= \int (1-u^2)^2\, du$$

$$= \int u^4 - 2u^2 + 1 \; du = \frac{u^5}{5} - \frac{2}{3}u^3 + u + C$$

$$= \frac{\sin^5 x}{5} - \tfrac{2}{3}\sin^3 x + \sin x + C.$$

2. $\int \sec^3 x \tan x \, dx.$ Let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$$\int \sec x \,.\, u \,.\, \frac{du}{dx}\, dx = \int \sqrt{1+u^2} \,.\, u \, du$$

Let $v = (1+u^2)$ $\frac{dv}{du} = 2u.$

$$\int \sqrt{v} \,.\, \tfrac{1}{2} \frac{dv}{du}\, du$$

$$= \int \tfrac{1}{2}\sqrt{v}\, dv = \tfrac{1}{3} v^{3/2} + C$$

$$= \tfrac{1}{3}\sec^3 x + C.$$

<u>or</u> Let $u = \sec x$ $\frac{du}{dx} = \sec x \tan x$

$$\int u^2 \frac{du}{dx} \,.\, dx = \int u^2 du$$

$$= u^3/3 + C$$

$$= \tfrac{1}{3}\sec^3 x + C.$$

Quicker but need more thought / knowledge

3 $\int \frac{x}{\sqrt{2x+1}}\,dx$ Let $u = 2x+1$

$\frac{du}{dx} = 2$

$x = \frac{1}{2}(u-1)$.

$= \int \frac{\frac{1}{2}(u-1)}{\sqrt{u}} \cdot \frac{1}{2} \frac{du}{dx}\,dx = \frac{1}{4}\int \sqrt{u} - u^{-1/2}\,du$

$= \frac{1}{4}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right) + C$

$= \frac{1}{6}(2x+1)^{3/2} - \frac{1}{2}(2x+1)^{1/2} + C$.

p 341 # 44

(a) $I = \int_{-1}^{1} \frac{1}{1+x^2}\,dx$

since $f(x) > 0$ $\int_a^b f(x) > 0$

(b) Let $u = \frac{1}{x}$.

$\frac{du}{dx} = -\frac{1}{x^2} = -u^2$

$\int \frac{1}{1+\frac{1}{u^2}}\,dx = \int \frac{u^2}{u^2+1}\,dx$

$= -\int \frac{1}{u^2+1} \frac{du}{dx} \cdot dx$

$x = 1 \Rightarrow u = 1$

$x = -1 \Rightarrow u = -1$

$-\int_{-1}^{1} \frac{1}{u^2+1}\,du$ So $I = -I$.

But as x goes from -1 to 1

$\frac{1}{x}$ goes from $-1 \to -\infty$ and $+\infty$ to 1

So cannot do this substitution.

Analysis Solutions 6

Problem (p308: 29-32). *If $f(x)$ is integrable on $[a, b]$ then $f(x)$ is continuous on $[a, b]$.*

It is the case that

$$0 < \int_{-1}^{1} \frac{\cos x}{\sqrt{1+x^2}}\,\mathrm{d}x$$

If the integral of f(x) over the interval $[a, b]$ is negative then $f(x) \leq 0$ on $[a, b]$

The function $f(x) = 0$ for $x < 0$ and $f(x) = x^2$ for $x > 0$ is integrable over every close interval.

Solution: False. Piecewise continuous functions are also integrable.

True. The integrand is positive in the domain of integration, and so the integral is strictly positive.

False. $\int_{-2}^{1} x\ \mathrm{d}x < 0$.

True. The function is continuous on every interval and so integrable. □

Problem (p320: 27-30). *There does not exists a differentiable function F such that $F'(x) = |x|$*

If $f(x)$ is continuous over an interval $[a, b]$ and if the integral of f over the interval is zero, then $f(x) = 0$ has at least one solution in that interval.

If $F(x), G(x)$ are the antiderivatives of $f(x), g(x)$ respectively, then

$$\int_a^b f(x)\mathrm{d}x = \int_a^b g(x)\mathrm{d}x$$

if and only if

$$G(a) + F(b) = F(a) + G(b)$$

If $f(x)$ is everywhere continuous, and $F(x) = \int_0^x f(t)\mathrm{d}t$ then the equation $F(x) = 0$ has at least one solution

Solution: False. Fundamental theorem guarantees that the integral of a continuous function is differentiable.

True. If $f(x) = 0$ had no solution, then by continuity either $f(x) > 0$ of $f(x) < 0$ in the interval, which would give a nonzero integral.

True. One direction is the fundamental theorem of calculus. For the other direction rewrite:

$$G(x) - F(x) = G(a) - F(a)$$

and so G and F differ by a constant, and thus have the same derivative $f(x) = g(x)$, so the integrals must also be equal.

True. $x = 0$ gives ,$F(0) = \int_0^0 f(x)\mathrm{d}x = 0$ and so is a solution.

□

MA1123 / Assignment 6

Q45) a) $\int_0^{10} h'(t)\,dt = \underbrace{h(10) - h(0)}$

The height growth in 10 years, after the child was born.

b) $\int_1^2 r'(t)\,dt = \underbrace{r(2) - r(1)}$

The radius difference at 2s and 1s

pp 335/336

Q15) $\boxed{\text{False}}$ i.e.: $f(x) = x$, $g(x) = 2 - x$ on $[0,4]$

Q16) $\boxed{\text{True}}$

$$(cf)_{ave} = \frac{1}{b-a}\int_a^b c f(x)\,dx = c \cdot \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= c\, f_{ave}$$

Q17) $\boxed{\text{True}}$

$$(f+g)_{aver} = \frac{1}{b-a}\int_a^b [f+g](x)\,dx$$

$$= \frac{1}{b-a}\left[\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right]$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b g(x)\,dx$$

$$= f_{aver} + g_{aver}$$

Q18) $\boxed{\text{False}}$ i.e. $f(x) = x$
$g(x) = x$ on $[0,2]$

$$f_{aver} = \frac{1}{2}\int_0^2 x\,dx = 1 = g_{aver}$$

$$\Rightarrow \boxed{f_{aver} \cdot g_{aver} = 1}$$

$$(f \cdot g)_{aver} = \frac{1}{2}\int_0^2 x^2\,dx = 4/3$$

Q24) $$V_{aver}^{0-10} = \frac{1}{10-0}\int_0^{10} 275.000 \sqrt{\frac{20}{t+20}}\,dt$$

$$= 27500 \left[2 \cdot \sqrt{20} \cdot \sqrt{t+20} \right]_0^{10}$$

$$\cong 247219 \ \$$$

247,500

p 342 # 45

(a) If $f(x)$ is odd $\quad f(-x) = -f(x)$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$u = -x$

$\frac{du}{dx} = -1$.

$$= \int_{a}^{0} f(-u)\cdot -du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0.$$

The algebraic areas cancel out

(b) If $f(x)$ is even $\quad f(-x) = f(x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{a}^{0} f(-x)\cdot -dx + \int_{0}^{a} f(x)\,dx.$$

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx.$$

The algebraic areas are the same.

p342 #46

$$\int_0^t f(t-x)g(x)\,dx,$$

Let $u = t-x$.

$\frac{du}{dx} = -1$.

$x = t-u$.

$x=0 \quad u=t$

$x=t \quad u=0$.

$$\int_t^0 f(u)\,g(\cancel{t}-u)\,.-1.\frac{du}{dx}\,dx$$

$$-\int_t^0 f(u)g(t-u)\,du = \int_0^t f(u)g(t-u)\,du.$$

#47. $$I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$$

Let $u = a-x$ $$\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}.-1.\,du = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx$$

$$I + I = \int_0^a \frac{f(x)+f(a-x)}{f(a-x)+f(x)}\,dx = \int_0^a 1.\,dx$$

$$= a$$

$\therefore \quad I = a/2$