MA1123 - Assignment 6

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a) Recalling the Newton's Method;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$, where r is root of $f(x)$ and $r \approx x_{n+1}$

In our problem; $f(x) = x^2 - a$, $f(\sqrt{a}) = 0$

$f'(x) = 2x$

By Newton's Method

$$\sqrt{a} \approx x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{x_n}{2} + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

b) For $\sqrt{10}$,

$$\sqrt{10} \approx x_{n+1} = \frac{1}{2}\left(x_n + \frac{10}{x_n}\right)$$

By very first approximation;

$$\sqrt{10} \approx x_1 = 3$$

$$\Rightarrow x_2 = \frac{1}{2}\left(3 + \frac{10}{3}\right) = 3.166667$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{10}{x_2}\right) = 3.162281$$

$\vdots$

$$\sqrt{10} \approx x_{n+1} = 3.162278$$

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a) By Newton's Method, $f(x) = 1/x - a$; $f'(x) = -1/x^2$

$$1/a \approx x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n + \frac{1/x_n - a}{1/x_n^2} = x_n + x_n - a x_n^2 = x_n (2 - a x_n)$$

b) For $1/17$, $a = 17$

$$1/17 \approx 1/20 = 0,05 \Rightarrow x_1 = 0,05$$

$$x_2 = x_1 (2 - 17 \cdot x_1) = 0,5 (2 - 17 \cdot 0,5) = 0,03$$

$$x_3 = x_2 (2 - 17 \cdot x_2) = 0,447$$

$\vdots$

$$1/17 \approx x_{n+1} = 0.5882\dot{3}\dot{5}$$

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a) (i) The total distance travelled in route (i) can be written as

$$d = d_L + d_C$$

where

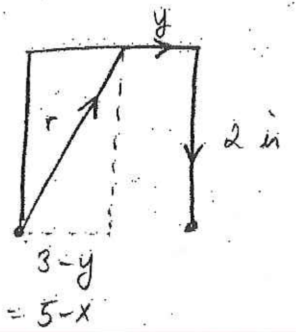
$d_L = 7$ in on linoleum and $d_C = 0$ in on carpet

$$\Rightarrow \quad t = \frac{d_L}{0.7} + \frac{d_C}{0.3} = 10\,s + 0\,s = 10\,s.$$

(iii) $d_L = 0$ in and $d_C = 3$ in

$$t = \frac{d_L}{0.7} + \frac{d_C}{0.3} = 0\,s + 10\,s = 10\,s$$

b)



$$y = x - 2$$

$$d_C = r = [(5-x)^2 + 4]^{1/2}$$

$$d_L = x$$

Range for y;

$$0 < y < 3$$

$\Rightarrow$ Range for x;

$$2 < x < 5$$

$$t = \frac{d_L}{0.7} + \frac{d_C}{0.3} = \frac{x}{0.7} + \frac{[(5-x)^2+4]^{1/2}}{0.3}$$

$$\frac{dt}{dx} = \frac{1}{0.7} - \frac{(5-x)}{[(5-x)^2+4]^{1/2}} \cdot \frac{1}{0.3}$$

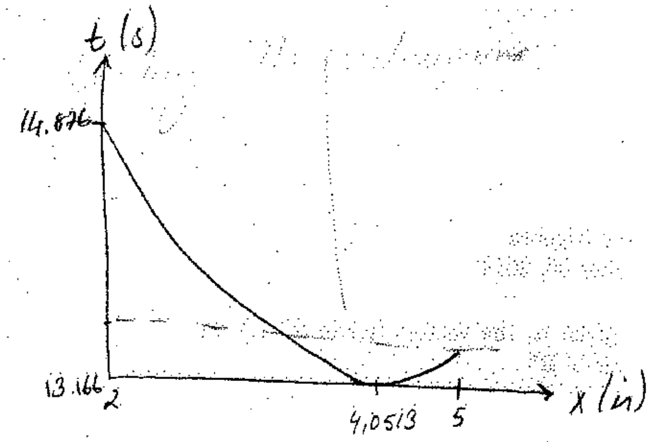
For the extremum point

$$3\,[(5-x)^2+4]^{1/2} - 7\,(5-x) = 0 \quad \Rightarrow \quad x = 4.0513$$

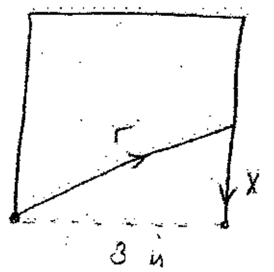
$$t(4.0513) = 13.166\ s$$

At end points;

$$t(2) = 14.876\ s \qquad t(5) = 13.816\ s$$



c)



$r = [x^2+9]^{1/2}$

$d_c = r = [x^2+9]^{1/2}$ and $d_l = x$

Range for x,

$0 < x < 2$

$$t = \frac{d_l}{0.7} + \frac{d_c}{0.3} = \frac{x}{0.7} + \frac{[x^2+9]^{1/2}}{0.3}$$

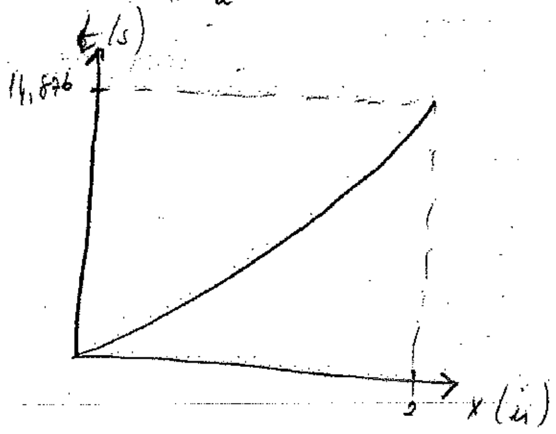
$$\frac{dt}{dx} = \frac{1}{0.7} + \frac{x}{0.3\,(x^2+9)^{1/2}}$$ for extremum $\frac{dt}{dx} = 0$

$\Rightarrow$ no root in the range!

Checking the limit cases

$\lim_{x \to 0^+} t(x) = 10\ s$

$\lim_{x \to 2^-} t(x) = 14.876\ s$



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11. This is false. Rolle's Theorem also need $f'(x)$ exists on (a, b) and even with that $f(a) = f(b)$, rather than $f(a) = f(b) = 0$ is an extension (that we did) of Rolle's Theorem

12. This is true. This is the velocity interpretation of the M.V. Th. see p 255.

13 This is false. It says $f'(x) = g'(x)$ $\Rightarrow$ $f(x) = g(x) + C$.

14. This is true. Thm 3.1.2 is proved using the M.V. Th.

22. $\forall$ $\frac{d}{dx}(3x^4 + x^2 - 4x) = 12x^3 + 2x - 4$.

Now $f(0) = f(1) = 0$, Hence by Rolle's $\exists$ c s.t $f'(c) = 0$ i.e. $12c^3 + 2c - 4 = 0$.

Note one can also use the Intermediate Value Theorem $f'(0) = -4$, $f'(1) = 10$ to get $f'(c) = 0$ some c in $(0, 1)$. But that does not use $\forall$ as asked. P.T.O.

8.2 Given $f'(x) = g(x)$ + $g'(x) = f(x)$ show

$$f^2(x) + g^2(x) = \text{constant}.$$

True $\Leftrightarrow$ $(f^2(x) + g^2(x))' = 0.$

$$(f^2(x) - g^2(x))' = 2f'(x)f(x) - 2g(x)g'(x)$$

$$= 2g(x)f(x) - 2g(x)f(x)$$

$$= 0$$

as required.

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Question 16

For a general cubic polynomial:

$$f(x) = ax^3 + bx^2 + cx + d, \qquad a \neq 0 \tag{1}$$

find the conditions on a, b, c and d to ensure that f is always increasing or decreasing on $(-\infty, \infty)$.

Solution

Condition for a function to be always increasing (decreasing) is that it its slope (derivative) is always positive (negative). Now we look at the derivative of our polynomial to find the conditions that a, b, c and d must obey for this to be true.

$$f'(x) = 3ax^2 + 2bx + c > 0, \tag{2}$$

For a quadratic to be always positive, there are no real solutions therefore we impose the condition that the discriminant of the equation ≤ 0.

$$(2b)^2 - 4(3a)(c) \leq 0 \implies b^2 \leq 3ac. \tag{3}$$

Once this condition is satisfied it is the sign of a that determines whether the cubic is always increasing or decreasing. If $a > 0$ then it is increasing as the derivative will be positive, and if $a < 0$ it is decreasing.

Also worth noting is that this solution does allow for the slope to be zero and if we are looking for the cubic to be strictly increasing (decreasing) then we use a $<$ symbol instead of a $\leq$ symbol.

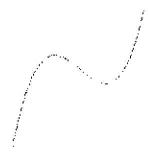
Question 18

(a) If f has a relative maximum at x_0, then $f(x_0)$ is the largest value that $f(x)$ can have.

(b) If the largest value for f on the interval (a, b) is at x_0, then f has a relative maximum at x_0.

(c) A function f has a relative extremum at each of its critical points.

Solution



(a) False. We can have a local maximum and the function can hit a local minimum and then increase beyond the original local max without ever having another (see figure).

(b) True. As the interval is an open interval then x_0 must be at some distinct point a finite distance from a or b. As it is the largest value on this interval then it is a relative max.

(c) False. x^3 has a critical point which isn't a relative extremum, it is a inflection point.

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Question 70

(a) $f(x) = |x-1|$ on $[-2,2]$.

Does not satisfy all the conditions as it is not everywhere differentiable on the interval.

(b) $f(x) = \frac{x+1}{x-1}$ on $[2,4]$. Does satisfy the conditions and $f'(x) = \frac{-2}{(x-1)^2}$.

$f(3) - f(2) = -1 \implies x = 1 \pm \sqrt{2}$

(c) $f(x) = \begin{cases} 3 - x^2 & \text{if } x \leq 1 \\ \frac{2}{x} & \text{if } x > 1 \end{cases}$ on $[0,2] \implies \frac{f(2)-f(0)}{2-0} = -1$

This function is continous and differentiable on the interval so we can apply the Mean-Value Theorem. And the only values of x for which $f'(c) = \frac{f(b)-f(a)}{b-a}$ are at $x = \frac{1}{2}$ and $x = \sqrt{2}$.