MA1123 - Solutions 5

Problem (10 - p234). *Find the point in the first quadrant of the curve $y = x^{-2}$ such that a rectangle with sides on the coordinate axes and a vertex P has the smallest possible perimeter.*

Solution: The perimeter function is:

$$\begin{aligned} f(x,y) =& 2x + 2y \\ f(x) =& 2x + \frac{2}{x^2} \\ \Rightarrow f'(x) =& 2 - \frac{4}{x^3} \\ \Rightarrow f''(x) =& \frac{4}{x^4} \end{aligned}$$

Since $f''(x) > 0$ in the first quadrant, the solution to $f'(x) = 0$ will give us the minumum, which is $P = (2^{1/3}, 2^{-2/3})$ □

Problem (14 - p234). *A wire of length 12cm can be bent into a circle, square, or cut into two pieces to make both. Find the length of wire used for the circle which a) minimises and b) maximises the enclosed area.*

Solution: Let the square have length a and the circle have radius r. Then the function we want to maximise is $f(a,r) = a^2 + \pi r^2$ subject to constraint $12 = 4a + 2\pi r$, so we substitute $a = 3 - \frac{\pi}{2}r$ to get the function:

$$\begin{aligned} f(r) =& (3 - \frac{\pi}{2}r)^2 + \pi r^2 \\ \Rightarrow f'(r) =& -\pi(3 - \frac{\pi}{2}r) + 2\pi r \\ \Rightarrow f''(r) =& \frac{\pi^2}{2} + 2\pi \end{aligned}$$

Since $f'' > 0$, the solution to $f'(r) = 0$ will be a local minimum, and since $f(r)$ is quadratic this is also the global minimum. This is at $r_0 = \frac{6}{4+\pi}$. To determine the global max we must look at the endpoints, $r = 0, a = 0$, which give $r_1 = 0, r_2 = 6/\pi$ when the wire is used entirely for the circle or the square. Now

$$\begin{aligned} f(r = 0) &= 9 \\ f(r = 6/\pi) &= \frac{36}{\pi} \end{aligned}$$

Since $\pi < 4$, we know $\frac{36}{\pi} > \frac{36}{4}$ and so the area is maximised when the wire is used entirely to form a circle. (In general this is the isoperimetric inequality).

□

Problem (30 - p234). *A closed cylindrical can is to have a surface area S. Show that the volume is maximised when the height is equal to the diameter of the base.*

$$a = 0 \Rightarrow r = \sqrt{\frac{S}{4\pi}}$$
$$r = 0 \Rightarrow a = \sqrt{\frac{S}{6}}$$

$$V(a=0) = \frac{4}{3}\pi\left(\frac{S}{4\pi}\right)^{3/2}$$
$$V(r=0) = \left(\frac{S}{6}\right)^{3/2}$$

Numerically checking gives that the volume is maximised when $a = 0$ and the surface area is entirely used for the sphere. (This is the 3d version of the isoperimetric inequality). $\square$

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Q 44)

a) $R(x) = x.p = x.(1000-x)$

b) $P(x) = R(x) - C(x) = x(1000-x) - (3000+20x)$

$= 980x - x^2 - 3000$

c) $P'(x) = 980 - 2x = 0 \Rightarrow x = 490$

d) $P(490) = 237100$ $P(0) = -3000$ $P(500) = 237,000$.

e) $P(p) = 980(1000-p) - (1000-p)^2 - 3000$

$$\frac{dP}{dp} = -980 + 2(1000-p) = 0$$

$\Rightarrow p = 510$

Q46) $C(t,n) = 15.t + 2,5\,n$, n: total gallon of diesel fuel

$t = x/v$, $n = x/_{10-0,07v}$, x: total distance.

$\Rightarrow C(x,v) = \frac{15x}{v} + 2,5\frac{x}{10-0,07v}$, total cost

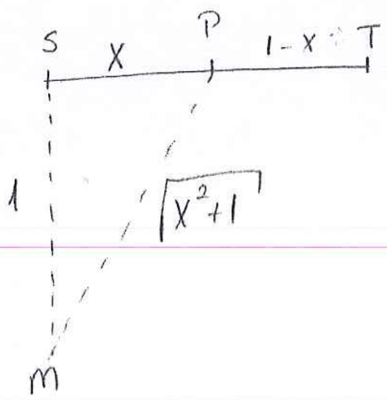
Cost per mile: $c(v) = 15/v + 2,5/_{10-0,07v}$

$$\frac{dc}{dv} = -15/v^2 + \frac{(0,07).2,5}{(10-0,07v)^2}$$

$\Rightarrow -15(10-0,07v)^2 + 2,5 . 0,07 v^2 = 0$

$\Rightarrow v = 56,1759$ miles/hour

Q56)



a) $T(x) = \dfrac{\sqrt{x^2+1}}{3} + \dfrac{1-x}{5}$

$T'(x) = \dfrac{1}{3}\dfrac{x}{\sqrt{x^2+1}} - \dfrac{1}{5} = 0$

$\Rightarrow x = 3/4$ critical point

$T(0) = 0.533 \qquad T(3/4) = 0.466$

$T(1) = 0,471$

$\Rightarrow x = 3/4$ mi $\rightarrow$ minimum time

b) $T(x) = \sqrt{x^2+1}/4 + 1-x/5$

$T'(x) = 1/4 \; x/\sqrt{x^2+1} - 1/5 = 0 \Rightarrow x = 4/3$ mi $\rightarrow$ out of range

$T(0) = 0,450 \qquad T(1) = 0,354$

$\Rightarrow x = 1$ mi is minimum

62) $t(x) = \dfrac{\sqrt{a^2+(c-x)^2}}{v_1} + \dfrac{\sqrt{b^2+x^2}}{v_1}$, v_1: speed of light, constant is uni. medium

$\dfrac{dt}{dx} = \dfrac{1}{v_1}\left[-\dfrac{(c-x)}{\sqrt{a^2+(c-x)^2}} + \dfrac{x}{\sqrt{b^2+x^2}}\right] = 0$

$\Rightarrow (c-x)^2 \cdot (b^2+x^2) = x^2\left(a^2+(c-x)^2\right)$

$\dfrac{b^2}{x^2} + 1 = \dfrac{a^2}{(c-x)^2} + 1 \Rightarrow \cot\theta_2 = \cot\theta_1$

for $0 < \theta_1, \theta_2 < 90 \Rightarrow \boxed{\theta_1 = \theta_2}$

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(a) $f(x) = \frac{1}{x} - a$.

$f'(x) = -\frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}}$$

$$= x_n + x_n - a x_n^2$$

$$= x_n(2 - a x_n).$$

(b) to approximate $\frac{1}{17}$

Let $x_1 = \frac{1}{20} = .05$

$x_2 = .05(2 - 17(.05))$

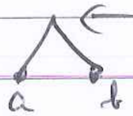
$= .0575$

$x_3 = .0575(2 - 17(.0575))$

$= .0588$ correct to 4 d.pl.

Note correct answer .0588235 - -

11 False Rolle's theorem also need $f'(x)$ to exist on (a, b).

Counterexample 

no tangent.

12 True. $f'(c) = \frac{f(b) - f(a)}{b - a}$ = Average rate of change

13 False $f(x) = 2x$ $g(x) = 3x$.

$f'(x) = 2$ $g'(x) = 3$

$f' - g' =$ constant

$f \neq g$.

14 True We used the M.V.Th. to prove Theorem 3.1.2.

15 $f(x) = \tan x$ $f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$

$\neq 0$ on $(0, \pi)$

But $f(x)$ is not diff at $\pi/2$.

16 $f(x) = x^{2/3}$ $a = -1$ $b = 8$.

$f(a) = 1$ $f(b) = 4$.

$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - (-1)} = \frac{1}{3}$.

$f'(x) = \frac{2}{3} x^{-1/3} = \frac{1}{3} \Rightarrow x^{-1/3} = \frac{1}{2}$

$\Rightarrow x^{1/3} = 2$

$\Rightarrow x = 8$

But $f(x)$ is not diff at $x = 0$.

17 (a) Two x-intercepts of $f(x)$

$\Rightarrow f(a) = f(b) = 0.$

By Rolle's Thm $\exists$ $a < c < b$

with $f'(c) = 0$

(b) $y = -x^2 + 4$ $y = 0$ at $x = \pm 2$

$\frac{dy}{dx} = -2x = 0$ at $x = 0.$

$y = \sin x$ $y = 0$ at $x = 0, \pi$

$\frac{dy}{dx} = \cos x = 0$ at $x = \pi/2$

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$f(x) = 3x^4 + x^2 - 4x$

$f(0) = 0$ $f(1) = 0$

By Rolles Thm $\exists$ c in $(0, 1)$

with $f'(c) = 0$ $f'(x) = 12x^3 + 2x - 4$

26 (a) $f(x) - f(y) = f'(c)\, x - y$

$|f(x) - f(y)| = |f'(c)||x - y|$

$\geq M|x - y|.$

(b) Let $f(x) = \tan x$ on $(-\frac{\pi}{2}, \pi/2)$

$f'(x) = \frac{1}{\cos^2 x} \geq 1$ on $(-\pi/2, \pi/2)$

$\therefore \ |\tan x - \tan y| \geqslant |x-y|$

all x, y in $(-\pi/2, \pi/2)$

(c) If y is in $(-\pi/2, \pi/2)$ so is $-y$.

and $\tan(-y) = \frac{\sin(-y)}{\cos(y)} = -\tan y$.

Hence $|\tan x - \tan(-y)| \geqslant |x-(-y)|$.

ie $|\tan x + \tan y| \geqslant |x+y|$.