Assignment 4.

1 To prove by induction that

$P(n)$ $(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + \cdots\cdots f_1 f_2 \cdots f_n'$

We know $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$. $P(2)$.

If we know $P(k)$ for $k \geq 2$

then $(f_1 f_2 \cdots f_{k+1})' = [(f_1 \cdots f_k) \cdot f_{k+1}]'$

$= (f_1 \cdots f_k)' f_{k+1} + (f_1 \cdots f_k) f_{k+1}'$ by $P(2)$

$= (f_1' \cdots f_k) f_{k+1} + \cdots (f_1 \cdots f_k') f_{k+1}$
$+ f_1 \cdots f_k f_{k+1}'$

and we have proven $P(k+1)$.

Hence $P(n)$ is true $\forall\, n \geq 2$.

2 (a) $y = \dfrac{x}{(x+2)^2}$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{(x+2)^2} - \dfrac{2x}{(x+2)^3}$

$$= \frac{x+2-2x}{(x+2)^3}$$

$$= \frac{-x+2}{(x+2)^3} = 0$$

$$\Rightarrow x = 2.$$

2

at $x = 0$ $\quad \frac{dy}{dx} > 0 \therefore f(x) \uparrow$.

$x = 3$ $\quad \frac{dy}{dx} < 0 \therefore f(x) \downarrow$

$x = 2$ local max.

$$\frac{d^2y}{dx^2} = \frac{-1}{(x+2)^3} - \frac{3(-x+2)}{(x+2)^4}$$

$$= \frac{-(x+2) + 3x - 6}{(x+2)^4}$$

$$= \frac{2x-8}{(x+2)^4} = 0 \quad \text{at } x = 4$$

4

at $x = 0$ $\quad \frac{d^2y}{dx^2} = -$ concave $\downarrow$

at $x = 5$ $\quad \frac{d^2y}{dx^2} = +$ concave up.

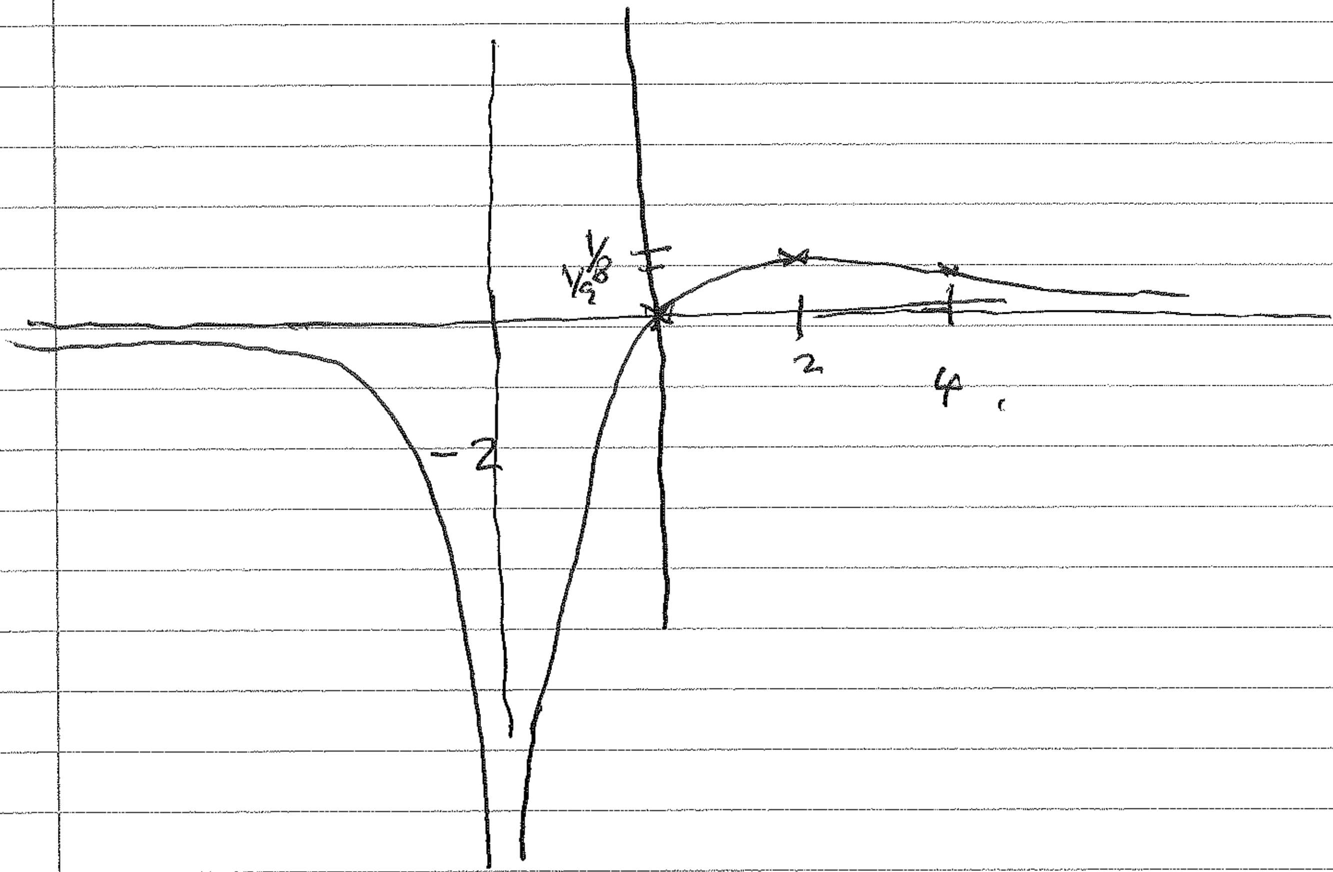
$x = -2$ is a vertical asymptote.

at $x \to -2^+$ $\quad y \to -\infty$

$x \to -2^-$ $\quad y \to -\infty$

$y = 0$ is a horizontal asymptote as $x \to \pm\infty$

A Rough Sketch.



2 (b) $y = \dfrac{x}{x^2-4}$

vertical asymptotes at $x = \pm 2$

$$\frac{dy}{dx} = \frac{1}{x^2-4} - \frac{x \cdot 2x}{(x^2-4)^2}$$

$$= \frac{x^2-4-2x^2}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2} = 0$$

$x^2 = -4$ Never

at $x = 0$ $\frac{dy}{dx} < 0$ $\therefore$ $f(x) \downarrow$.

at $x = 3$ $\frac{dy}{dx} < 0$ $\therefore$ $f(x) \downarrow$

at $x = -3$ $\frac{dy}{dx} < 0$ $\therefore$ $f(x) \downarrow$.

$$\frac{d^2y}{dx^2} = \frac{-2x}{(x^2-4)^2} + \frac{(-x^2-4) \cdot -2}{(x^2-4)^3} \cdot 2x$$

$$= \frac{-2x(x^2-4) + 4x(x^2+4)}{(x^2-4)^3}$$

$$= \frac{-2x^3 + 8x + 4x^3 + 16x}{(x^2-4)^3}$$

$$= \frac{2x^3 + 24x}{(x^2-4)^3} = 0 \qquad \frac{2x(x^2+12)}{(x^2-4)^3} = 0$$

Critical pts for $\frac{dy}{dx}$ $\Rightarrow$ $x = 0$ and $x = \pm 2$

$-2 \qquad 0 \qquad 2$

$x = -3 \qquad \frac{d^2y}{dx^2} = \frac{-}{+} = - \qquad$ concave ↓

$x = -1 \qquad \frac{d^2y}{dx^2} = \frac{-}{-} = + \qquad$ concave ↑

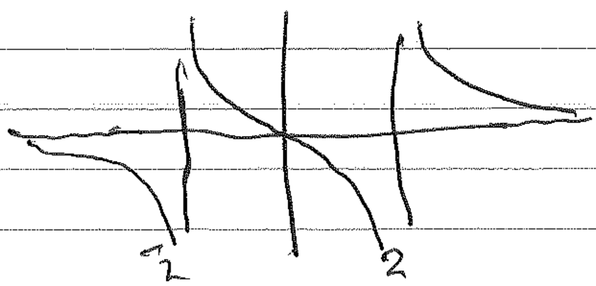
$x = 1 \qquad \frac{d^2y}{dx^2} = \frac{+}{-} = - \qquad$ concave ↓

$x = +3 \qquad \frac{dy}{dx^2} = \frac{+}{+} = + \qquad$ concave ↑

$x = 0$ is a point of inflection.

As $x \to \pm\infty$, $y \to 0$

Sketch



MA 1123 / Assign. 4

pp. 222, 223

Q12) $f(x) = \sqrt[3]{(x^2+x)^2}$ $[-2, 3]$

$$f'(x) = \frac{2}{3} \cdot (x^2+x)^{-1/3} \cdot (2x+1) = \frac{2}{3} \frac{(2x+1)}{\sqrt[3]{(x^2+x)}}$$

i) $f(x_0) = \frac{2}{3} \frac{(2x_0+1)}{\sqrt[3]{x_0^2+x_0}} = 0 \Rightarrow x_0 = -1/2$ relative extremum point

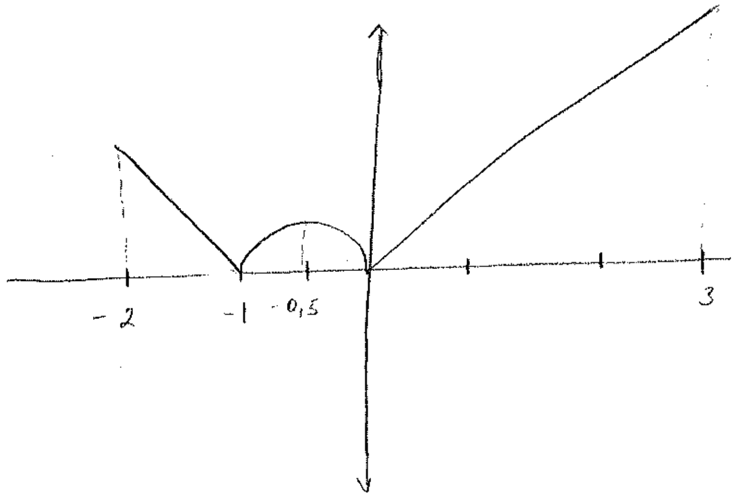
ii) for $x^2+x=0 \Rightarrow x=0$ and -1 are also critical points, since $f'(0)$ and $f'(-1)$ are not defined.

However, $\lim_{x\to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0) = 0$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x) = f(-1) = 0$

iii) Let us check the end points

$f(-2) = \sqrt[3]{4}$ and $f(3) = \sqrt[3]{144}$



Abs. minimum $= 0$ at $x = -1$ and 0

Abs. maximum $= \sqrt[3]{144}$ at $x = 3$

Analysis: Solutions 4

State whether the following are true or false.

Problem 1: *If f is decreasing on $[0,2]$ then $f(0) > f(1) > f(2)$.*

Solution: True. Follows from definition of (strictly) decreasing. $\square$

Problem 2: $f'(1) > 0$, *the f is increasing on* $[0,2]$.

Solution: False. The derivative at a single point does not tell us enough to say anything about it's behaviour on the whole interval. For example, $f'(0.5) < 0$ can also be true, which contradicts the assumption. $\square$

Problem 3: *If f is increasing on $[0,2]$, then $f'(1) > 0$.*

Solution: True. Follows from the derivative: $f(1+h) - f(1) > 0$. $\square$

Problem 4: *If f' is increasing on $[0,1]$ and f' is decreasing on $[1,2]$, then f has an inflection point at $x = 1$.*

Solution: False. We've assumed that f' exists everywhere but f'' may not exist everywhere. In particular, $f''(1)$ may not exist, in which case it is not an inflection point. $\square$

Determine whether the statements are true or false, and find counterexamples if false.

Problem 5: *if f and g are increasing on an interval, then so is $f + g$.*

Solution: True. $(f+g)' = f' + g' > 0$ if both $f', g' > 0$. $\square$

Problem 6: *If f and g are increasing on an interval, then so is $f \cdot g$.*

Solution: False. Let $f(x) = x$ and $g(x) = x$. Then $fg = x^2$ which is decreasing $\forall x < 0$. $\square$

Problem 7: *Prove that a general cubic polynomial has exactly one inflection point.*

Solution: Let $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$. Then

$$f''(x) = 6ax + 2b$$

Which has the unique solution $x = -\frac{b}{3a}$ since $a \neq 0$. $\square$

Problem 8: *Prove that if a cubic polynomial has three x intercepts, that the inflection point occurs at the average of these three.*

Solution: If the polynomial has solutions x_1, x_2, x_3, we can write

$$\begin{aligned} ax^3 + bx^2 + cx + d =& a(x - x_1)(x - x_2)(x - x_3) \\ =& ax^3 - ax^2(x_1 + x_2 + x_3) + O(x) \end{aligned}$$

where $O''(x) = 0$. We can compare coefficients, and use the previous answer

$$\begin{aligned} b =& -a(x_1 + x_2 + x_3) \\ \Rightarrow x_{inflection} =& \frac{x_1 + x_2 + x_3}{3} \end{aligned}$$

$\square$

Problem 9: *Use the previous result to find the inflection point of* $f(x) = x^3 - 3x^2 + 2x$

Solution:

$$f(x) = x(x^2 - 3x + 2) = x(x-2)(x-1)$$

And so $x_1 = 0, x_2 = 1, x_3 = 2$. Thus,

$$x_{inflection} = 1$$

.

Using $f''(x) = 6x - 6$ gives the POI at $x = 1$ also. Also, $f''(0) = -6 < 0$ so $f(x)$ is concave down for $x < 1$ and concave up for $x > 1$. $\square$