MA1123 Solution 3

Problem 1: *Find the derivative of $f(x) = \sqrt[3]{x}$*

Solution: The difference quotient is $\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$, and we want to take $\lim_{h\to 0}$. Making use of the difference of two cubes

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2+ab+b^2) \\ \Rightarrow (x+h) - x &= ((x+h)^{1/3} - x^{1/3})\left((x+h)^{2/3} + (x(x+h))^{1/3} + (x+h)h^{2/3}\right) \\ \Rightarrow \frac{(x+h)^{1/3} - x^{1/3}}{h} &= \frac{1}{(x+h)^{2/3} + (x(x+h))^{1/3} + (x+h)^{2/3}} \end{aligned}$$

We can now take $\lim_{h\to 0}$ of each side. The left hand side is by definition the derivative, and the right hand side is a well-behaved function as $h \to 0$ so we can substitute $h = 0$.

if $x \neq 0$ and by cont. of $x^{2/3}$ + $x^{1/3}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt[3]{x} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}$$

□

Problem 2: *Find the derivative of $f(x) = cos(x)$*

Solution:

Considering the finite difference

$$\begin{aligned} f(x+h) - f(x) &= \cos(x+h) - \cos(x) \\ &= \cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x) \\ &= \cos(x)(\cos(h) - 1) - \sin(x)\sin(h) \end{aligned}$$

Now divide by h to get the difference quotient, and take $h \to 0$, giving two terms. The first of which

$$\begin{aligned} &\lim_{h\to 0} \frac{\cos(x)(\cos(h) - 1)}{h} \\ &= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} \\ &= 0. \end{aligned}$$

Now using the limit $\frac{sin(x)}{x} \to 1$ as $x \to 0$, we get

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= -\sin(x) \lim_{x\to 0} \frac{\sin(h)}{h} \\ &= -\sin(x) \end{aligned}$$

□

Solutions

MA1123 / Assignment 3.

Q3)

a) $y = \sec x = \frac{1}{\cos x} \Rightarrow dy = \left[\frac{-(-\sin x)}{\cos^2 x}\right] dx \Rightarrow \frac{dy}{dx} = \frac{\sin x}{\cos^2 x}$

b) $y = \sin\sqrt{\cos(x^3+2x+1)}$

$$dy = \left\{\left[\cos\sqrt{\cos(x^3+2x+1)}\right]\cdot\left[\frac{1}{2}\left(\cos(x^3+2x+1)\right)^{-1/2}\right]\cdot \left[-\sin(x^3+2x+1)\right]\left[3x^2+2\right]\right\} dx$$

c) $x^2y^3 + \cos(xy) = xy$

$$2xy^3\,dx + 3x^2y^2\,dy - y\sin(xy)\,dx - x\sin(xy)\,dy = y\,dx + x\,dy$$

$$\left[3x^2y^2 - x\sin(xy) - x\right]dy = \left[-2xy^3 + y\sin(xy) + y\right]dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xy^3 + y\sin(xy) + y}{3x^2y^2 - x\sin(xy) - x}$$

d) $y = x^2\cos x\sin x$

$$dy = \left[2x\cos x\sin x - x^2\sin^2 x + x^2\cos^2 x\right]dx$$

$$= \left[x\sin 2x + x^2\cos 2x\right]dx$$

Q4

(a) $$\lim_{x \to \infty} \frac{3x^3 + 2x + 1}{4x^3 + 2x^2 + 2}$$

Divide above and below by the highest power of x in the denominator ie x^3

$$\lim_{x \to \infty} \frac{3 + 2/x^2 + 1/x^3}{4 + 2/x + 2/x^3} \to \frac{3 + 0 + 0}{4 + 0 + 0} = \frac{3}{4}.$$

(b) $$\lim_{x \to \infty} \frac{3x^3 + 2x + 1}{4x^4 + 2x^2 - 7} = \lim_{x \to \infty} \frac{3/x + 2/x^3 + 1/x^4}{4 + 2/x^2 - 7/x^4} = \frac{0}{4}$$
$$= 0.$$

(c) $\lim_{x \to 0} x \sin \frac{1}{x}$. $\quad -1 \leq \sin \frac{1}{x} \leq 1$

and $x \to 0$

So $$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

(d) $$\lim_{x \to 0} \frac{\sin 5x}{\sin 7x} = \lim_{x \to 0} \frac{\sin 5x}{5x} . 5x . \frac{7x}{\sin 7x} . \frac{1}{7x}$$

$$= \lim_{x \to 0} \frac{\sin 5x}{5x} . \frac{5}{7} . \lim_{x \to 0} \frac{7x}{\sin 7x}$$

But $5x \to 0 \iff x \to 0$

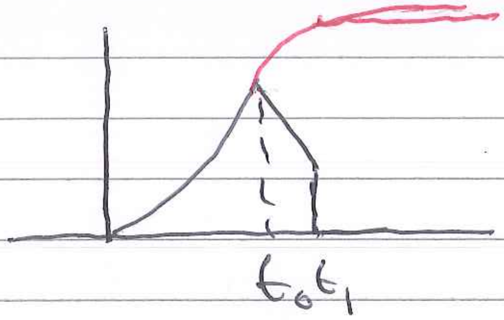
$$= 1 . 5/7 . 1 = 5/7.$$

p120

#5 velocity $= \frac{ds}{dt}$ = constant.

$\Rightarrow s = at + b$ — a line.

#6



#19 True put $x = 1 + h$ as we have done many times before

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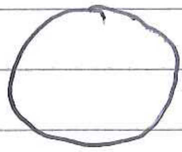
#12



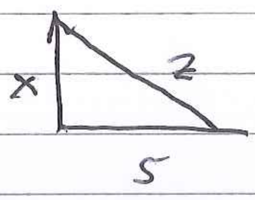
$\frac{dr}{dt} = 3$ ft/sec.

$A = \pi r^2$

$\frac{dA}{dt} = \pi . 2r \frac{dr}{dt}$.

after 10 secs $r = 30$ ft $\therefore$ $\boxed{\frac{dA}{dt} = 60\pi . 3 = 180\pi}$.

#20



$x^2 + 25 = z^2$

$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

when $x = 4$, $z = \sqrt{41}$, $\frac{dz}{dt} = 2000$ m/h

$\frac{dx}{dt} = \frac{\sqrt{41}}{4} . 2000 = \boxed{500\sqrt{41}}$