MA1123 Solution 2

Problem 1: *Prove that* $\lim_{x\to 0}\sin(\frac{1}{x})$ *does not exist.*

Solution: Assume $\exists L$ with $\lim_{x\to 0} sin(\frac{1}{x}) = L$. Then $\forall\epsilon > 0\exists\delta, 0 < |x-a| < \delta \Rightarrow |\sin(\frac{1}{x})-L| < \epsilon$. To reach a contradiction, we choose an ϵ and show that no δ will satisfy the definition. Since $\sin(x)$ oscillates between -1 and 1 for large x, $\epsilon = 1/2$ will suffice.

Now, choose two values $x_1, x_2 < \delta$ with $\sin(1/x_1) = 1, \sin(1/x_2) = -1$. There will always be sufficiently large n with $x_1 = \frac{1}{2\pi n+\pi/2}$ and $x_2 = \frac{1}{2\pi n-\pi/2}$ such that $x_1, x_2 < \delta$. Then for each of these values of x we must have:

$$
\begin{aligned}
|\sin\left(\frac{1}{x_1}\right) - L| < \frac{1}{2} \;\;&\&\;\; |\sin\left(\frac{1}{x_2}\right) - L| < \frac{1}{2}\\
\Rightarrow |1-L| < \epsilon \;\;&\&\;\; |-1-L| < \epsilon\\
\Rightarrow 1-\epsilon < L < 1+\epsilon \;\;&\&\;\; -1-\epsilon < L < \epsilon - 1\\
\Rightarrow -\frac{3}{2} < L < -\frac{1}{2} \;\;&\&\;\; \frac{1}{2} < L < \frac{3}{2}.
\end{aligned}
$$

Since these inequalities cannot be simultaneously satisfied, we reach a contradiction, and conclude that the limit does not exist.

□

Problem 2: *Prove that* $\lim_{x\to a}\sum_i^n f_i(x) = \sum_i^n L_i$, *if* $\lim_{x\to a} f_i(x) = L_i, \forall i \in \{1, ..., n\}$, *using induction.*

Solution: Our base case is $n = 2$, which is just

$$\lim_{x\to a} f_1(x) + f_2(x) = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x).$$

This is the statement that the limit of the sums is the sum of the limits and was proved in class.

Now assume that the proposition holds for $n = k$. Then for $n = k+1$ we have

$$\lim_{x\to a}\sum_i^{k+1} f_i(x) = \lim_{x\to a}\left(f_{k+1}(x) + \sum_i^k f_i(x)\right).$$

This is now the same form as $\lim(f+g) = (\lim f) + (\lim g)$ with $f(x) = f_{k+1}(x)$ and $g(x) = \sum_i^k f_i(x)$. Since, by our inductive hypothesis $\lim_{x\to a} g(x) = \sum_i^k L_i$, and again invoking "the limit of the sum is the sum of the limits", we have

$$\lim_{x\to a}\sum_i^{k+1} f_i(x) = L_{k+1} + \sum_i^k L_i = \sum_i^{k+1} L_i$$

□

3. If $\lim_{x\to a} f(x) = L$ and $L \neq 0$, then $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L}$

We want $\left|\frac{1}{f(x)} - \frac{1}{L}\right| < \epsilon$ if $0 < |x-a| < \delta$

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| = \left|\frac{f(x) - L}{f(x) \cdot L}\right| = \frac{|f(x) - L|}{|L|} \cdot \frac{1}{|f(x)|}.$$

we must find M such that $\frac{1}{|f(x)|} < M$

i.e. $|f(x)| > \frac{1}{M} = N$ say.

Now if $L \neq 0$, we can find δ_1 such that

$0 < |x-a| < \delta_1 \to |f(x) - L| < \frac{|L|}{2}$.

i.e. $L - \frac{|L|}{2} < f(x) < L + \frac{|L|}{2}$.

if $L > 0$ $\quad L - \frac{|L|}{2} < f(x) \Rightarrow \frac{|L|}{2} < f(x) = |f(x)|$.

if $L < 0$ $\quad f(x) < L + \frac{|L|}{2} \to f(x) < -\frac{|L|}{2}$

$\to |f(x)| > \frac{|L|}{2}$.

So $\frac{1}{M} = \frac{|L|}{2}$ $\qquad M = \frac{2}{|L|}$.

So $\frac{|f(x) - L|}{|L|} \cdot \frac{1}{|f(x)|} < \frac{|f(x) - L|}{|L|} \cdot \frac{2}{|L|}$ if $0 < |x-a| < \delta_1$

Now given ϵ $\exists\, \delta_2$ such that $0 < |x-a| < \delta_1$

$\Rightarrow \frac{|f(x) - L|\, 2}{|L|^2} < \epsilon$.

So if $0 < |x-a| < \delta = \min(\delta_1, \delta_2)$

then $\left|\frac{1}{f(x)} - \frac{1}{L}\right| < \epsilon$

MA1123, Assignment 2

Q2) $\lim_{x \to 1/4} 1/x^2 = 16 \Rightarrow \forall \varepsilon > 0$ and $\exists \delta >$ s.t. $0 < |x - 1/4| < \delta$

$\Rightarrow |1/x^2 - 16| < \varepsilon$

$$|1/x^2 - 16| = |x - 1/4| \, \left| \frac{4 + 16x}{x^2} \right| < \varepsilon$$

Let $|x - 1/4| < \delta_1 = 1/8$

$\Rightarrow 1/8 < x < 3/8 \Rightarrow x = 1/8$ for max

$|x - 1/4| \, 384 < \varepsilon \Rightarrow |x - 1/4| < \varepsilon/384 = \delta_2$

$\delta = \min \{ \delta_1, \delta_2 \}$

Q5) For continuity at $x = 0$,

$$\lim_{x \to 0^+} (3x + b) = \lim_{x \to 0^-} (-7x) = f(0) = 0$$

$\Rightarrow b = 0$