MA1123 Solution 1

Problem 1: *In the tutorial we considered the product, sum, and composition of two even functions. Repeat these considerations for two odd functions, along with one odd and one even function*

Solution:

Both Odd:

Let f and g be two odd functions, i.e. $f(-x) = -f(x)$, $g(-x) = -g(x)\ \forall x \in \mathbb{R}$. We want to see what happens to the value of the product, sum and composition of these two function when we change the sign of the input. For the sum we have:

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x) \\ &= -f(x) - g(x) \\ &= -(f(x) + g(x)) \\ &= -(f+g)(x)\end{aligned}$$

and so $f + g$ is also an odd function.

The product is:

$$\begin{aligned}(f \cdot g)(-x) &= f(-x)g(-x) \\ &= (-f(x))(-g(x)) \\ &= f(x)g(x) \\ &= (fg)(x)\end{aligned}$$

and so fg is an even function.

The composition function $f \circ g$ is:

$$\begin{aligned}(f \circ g)(-x) &= f(g(-x)) \\ &= f(-g(x)) \\ &= -f(g(x)) \\ &= -(f \circ g)(x)\end{aligned}$$

which is therfore odd.

One odd one even:

Let f be an odd function and g be an even function. We'll start with the product fg:

$$\begin{aligned}(f \cdot g)(-x) &= f(-x)g(-x) \\ &= (-f(x))(g(x)) \\ &= -(f \cdot g)(x)\end{aligned}$$

and so the product of an odd and even function is odd. The composition is $f \circ g$:

$$\begin{aligned}(f \circ g)(-x) &= f(g(-x)) \\ &= f(g(x)) \\ &= (f \circ g)(x)\end{aligned}$$

which is even. Because composition of functions is not a commutative operation, that is $f \circ g \neq g \circ f$, we also need to check the compoision $g \circ f$:

$$\begin{aligned}(g \circ f)(-x) &= g(f(-x)) \\ &= g(-f(x)) \\ &= g(f(x)) \\ &= (g \circ f)(x)\end{aligned}$$

which is also even. Therefore the composition of an even and odd function is even.

Finally, the sum:

$$\begin{aligned}(f+g)(-x) &= f(-x)+g(-x)\\ &= -f(x)+g(x).\end{aligned}$$

We cannot say whether this is even or odd. For example, let $f(x) = x$ and $g(x) = 1$, then $(f+g)(x) = x+1$ is neither even nor odd. We can go slightly further and check for which functions this expression is even or odd. Assuming it's even

$$\begin{aligned}&(f+g)(-x) = (f+g)(x)\\ &-f(x)+g(x) = f(x)+g(x)\\ &\Rightarrow f(x) = 0\end{aligned}$$

and if it's odd

$$\begin{aligned}&(f+g)(-x) = -(f+g)(x)\\ &-f(x)+g(x) = -f(x)-g(x)\\ &\Rightarrow g(x) = 0\end{aligned}$$

So in general, the sum of an odd and an even function is neither odd or even, and is only odd or even if one the functions is zero. (Recall from the tutorial that the zero function is the only function that is both odd and even).

□

MA1123, 2015-2014, Homework 1

Q2 Let us begin with the definition of distance function. Any real valued function, d defined on the metric space X is called *distance function*, if it satisfies the following properties:

$$\forall a,b,c \in X,$$

$$d(a,b) \geq 0 (Non-negativity) \tag{1}$$

$$d(a,b) = 0 \iff a = b (Identity of indiscernible) \tag{2}$$

$$d(a,b) = d(b,a) (Symmetry) \tag{3}$$

$$d(a,b) \leq d(a,c) + d(c,b) (Triangle inequality) \tag{4}$$

Let us show that $d = |a-b|$ has these properties.

1. By definition of the absolute function, $|a-b| \geq 0$.
2. By properties of the absolute function, $|a-b| = 0 \iff a-b=0 \;\; and \;\; a = b$.
3. By properties of the absolute function, $|a-b| = |b-a|$.
4. By properties of the absolute function (proved in the classroom), $|a-b| = |(a-c)-(b-c)| \leq |a-c| + |b-c| = |a-c| + |c-b|$

Let us check case by case that $|a-b|$ is distance in real line.

- For a,b > 0 and a $>$ b, $|a-b| = a-b$ (Chose two point as a and b on the positive real axis, and show the part $|a-b|$ represents.) .
- For a > 0 and $b <$, $|a-b| = a-b = a+|b|$ (Chose two point as a on the positive and b on the negative real axis, and show the part $|a-b|$ represents.).

- For a, b < 0 and $|b| \leq |a|$, $|a-b| = b-a = |a|-|b|$(Chose two point as a and b on the negative real axis, and show the part $|a-b|$ represents.)

Q3

A:Let us assume that f *and* $g : X \to X$ are both 1-1 (injective).

- $f+g$ can be either injective or not. (Draw some example sets and relations for both case.)
- $f.g$ can be either injective or not. (Draw some example sets and relations for both case).
- $f \circ g$ are injective.
  Let us assume f(g(a))=f(g(b)). Since f is injective, g(a)=g(b). Since g is also injective, then a=b.

B:Let us assume that f *and* $g : X \to X$ are both onto (surjective).

- $f+g$ can be either surjective or not. (Draw some example sets and relations for both case.)
- f.g can be either surjective or not. (Draw some example sets and relations for both case).
- $f \circ g$ are surjective.
  Since g is onto, there is an element b such as g(b)=a for any element a in X. Similarly, since f is onto, there i an element c in X such as f(c)=b. So there is an element c in X for any a such as (f ∘ g) (c)=a

C:Let us assume that f *and* $g : X \to X$ are both 1-1 and onto (bijective).

- $f+g$ can be either bijective or not. (Draw some example sets and relations for both case.)
- $f.g$ can be either bijective or not. (Draw some example sets and relations for both case).
- f∘g is bijective from A and B.

Assignment 1 MA1123

Q4. Prove, from the definition, that

$$\lim_{x \to 2} x^3 + 2x^2 - 1 = 15.$$

Want $| x^3 + 2x^2 - 1 - 15 | < \epsilon$ if $0 < |x - 2| < \delta$

$$| x^3 + 2x^2 - 16 | = | (x-2)(x^2 + 4x + 8) |$$

$$= | x - 2 | | x^2 + 4x + 8 |$$

$$\begin{array}{r|l} & x^2 + 4x + 8 \\ \hline x - 2 & x^3 + 2x^2 - 16 \\ & x^3 - 2x^2 \\ \hline & 4x^2 - 16 \\ & 4x^2 - 8x \\ \hline & 8x - 16 \end{array}$$

Let $|x - 2| < 1$

then $1 < x < 3$

& $| x^2 + 4x + 8 | < 29$.

So if $|x - 2| < 1$

then $| x^3 + 2x^2 - 16 | < |x - 2| . 29$.

So given any $\epsilon > 0$
Let $\delta = \min(1, \epsilon/29)$

then $| x^3 + 2x^2 - 16 | < \delta . 29 \le \epsilon$

ie $| x^3 + 2x^2 - 1 - 15 | < \epsilon$

Q5. $| 2x - 3 | < 4$

means $-4 < 2x - 3 < 4$

$-4 + 3 < 2x < 4 + 3$

$-\frac{1}{2} < x < \frac{7}{2}$ since $2 > 0$.