MA1123 Assignment 10.

1 Solve $(D+2)^3 y = 0$ in three steps.

Let $z = (D+2)^2 y$. Want $(D+2)z = 0$

know $\Rightarrow$ $z = A_1 e^{-2x}$.

$$(D+2)^2 y = A_1 e^{-2x}.$$

Let $w = (D+2)y$ Want $(D+2)w = A_1 e^{-2x}$

$$\frac{dw}{dx} + 2w = A_1 e^{-2x}.$$

This is linear, first order. Use an integrating factor $\mu = e^{\int P(x)\,dx}$.

$P(x) = 2$ $\int P(x)\,dx = 2x$ $\mu = e^{2x}$.

$$e^{2x}\frac{dw}{dx} + 2e^{2x}w = A_1$$

$$\frac{d}{dx}(e^{2x}w) = A_1$$

$$e^{2x}w = A_1 x + A_2.$$

$$w = A_1 x e^{-2x} + A_2 e^{-2x}.$$

$(D+2)y = w$ - linear, first order

$\frac{dy}{dx} + 2y = w$. Int. factor $= e^{2x}$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = A_1 x + A_2.$$

$$\frac{d}{dx}(e^{2x} y) = A_1 x + A_2$$

$$e^{2x} y = \frac{A_1}{2} x^2 + A_2 x + A_3.$$

The solution are $y = \frac{A_1}{2} x^2 e^{-2x} + A_2 x e^{-2x} + A_3 e^{-2x}$

2. Solve $(D+2)^3 y = 3x$

$$\Rightarrow D^2 (D+2)^3 y = 0.$$

$$\Rightarrow y = A_1 x^2 e^{-2x} + A_2 x e^{-2x} + A_3 e^{-2x} + A_4 x + A_5.$$

Put $(D+2)^3 y = 3x.$

Note $(D+2)^3 (A_1 x^2 e^{-2x} + A_2 x e^{-2x} + A_3 e^{-2x}) = 0$

So only need $(D+2)^3 (A_4 x + A_5) = 3x$

D^3	0
$6D^2$	0
$12D$	$12A_4$
8	$8A_4 x + 8A_5$

$$8A_4 x + 12A_4 + 8A_5 = 3x$$

$8A_4 = 3 \qquad A_4 = 3/8$

$12A_4 + 8A_5 = 0. \qquad A_5 = -9/16$

Soln $y = A_1 x^2 e^{-2x} + A_2 x e^{-2x} + A_3 e^{-2x} + 3/8\, x - 9/16.$

3 Solve $(D+2)(D+3)y = 2\cos 3x.$

Now $2\cos 3x$ is a solution to

$$(D^2+9)y = 0.$$

So $(D^2+9)(D+2)(D+3)y = 0$

$\Rightarrow$ $y = A_1 e^{-2x} + A_2 e^{-3x} + A_3 \cos 3x + A_4 \sin 3x$

Put $(D+2)(D+3)y = 2\cos 3x.$

$A_1 e^{-2x} + A_2 e^{-3x} \longrightarrow 0.$

Only need $(D+2)(D+3)(A_3 \cos 3x + A_4 \sin 3x)$

$D^2 \quad -9A_3 \cos 3x - 9A_4 \sin 3x.$

$5D \quad 15(-A_3 \sin 3x + A_4 \cos 3x).$

$6 \quad 6A_3 \cos 3x + 6A_4 \sin 3x$

$(-9A_3 + 15A_4 + 6A_3)\cos 3x + (-9A_4 - 15A_3 + 6A_4)\sin 3x.$

$= 2\cos 3x.$

$-3A_3 + 15A_4 = 2$

$-15A_3 - 3A_4 = 0.$

$-75A_3 - 15A_4 = 0.$

$-78A_3 = 2 \qquad A_3 = -\frac{2}{78} = -\frac{1}{39}.$

$A_4 = 5/39.$

So the solutions are

$$y = A_1 e^{-2x} + A_2 e^{-3x} - \frac{1}{39}\cos 3x + 5/_{39} \sin 3x$$

p 576 #36 40% decays in 5 years

so 60% remains.

$$y(t) = y(0) e^{kt}$$

$t = 5$ $\quad \frac{6}{10} y(0) = y(0) e^{k.5}$

$\ln \frac{6}{10} = 5k \qquad k = \frac{\ln(.6)}{5}.$

Want t_0 when $y(t_0) = \frac{1}{2} y(0)$

$\frac{1}{2} y(0) = y(0) e^{kt_0}$

$\ln \frac{1}{2} = k t_0 \qquad t_0 = \frac{\ln .5}{k}$

$t_0 = 5 \frac{\ln .5}{\ln .6} = 6.7845$

<u>Note</u> answer must be > 5.

(b) $100 e^{kt} = 100 e^{.022t}$

p 597 (a) r = annual interest rate.

$\frac{r}{n}$ = % interest per period.

After 1 period $\quad P + P\frac{r}{n}$

capital + interest.

$= P\left(1+\frac{r}{n}\right)^1$

after 2nd Period

$$P + P\frac{r}{n} + \left(P + P\frac{r}{n}\right)\frac{r}{n}.$$

capital + interest.

$$P(1+r/n) + P(1+r/n)r/n.$$

$$= P(1+r/n)(1+r/n)$$

$$= P(1+r/n)^2$$

etc.

after t years = nt periods

$$= P(1+r/n)^{nt}.$$

(b) $(1+r/n)^{nt} = (1+r/n)^{n/r \cdot rt}$

$n \to \infty$ $\quad r/n \to 0$

$$\to e^{rt}.$$

Amount $\to Pe^{rt}$

(c) $A(t) = Pe^{rt}$

$$\frac{dA}{dt} = rPe^{rt} = rA(t).$$

So $\frac{dA}{dt} \propto A$.

p 592 #5

$$(x^2+1)\frac{dy}{dx} + xy = 0.$$

Linear, first order - put in standard form $\frac{dy}{dx} + P(x)y = Q(x)$,

$$\frac{dy}{dx} + \frac{x}{x^2+1}\,y = 0$$

$P(x) = \frac{x}{x^2+1}$ $\int P(x)\,dx = \frac{1}{2}\ln(x^2+1).$

$$\mu = e^{\ln\sqrt{x^2+1}} = \sqrt{x^2+1}.$$

$$\sqrt{x^2+1}\frac{dy}{dx} + \frac{x}{\sqrt{x^2+1}}\,y = 0.$$

$$\frac{d}{dx}\left(\sqrt{x^2+1}\;y\right) = 0.$$

$$\sqrt{x^2+1}\;y = A.$$

$$y = \frac{A}{\sqrt{x^2+1}}$$

#8

$$x\frac{dy}{dx} - y = x^2$$

Linear, first order, put in standard form

$$\frac{dy}{dx} - \frac{1}{x}y = x.$$

$P(x) = -\frac{1}{x}$ $\int P(x)\,dx = -\ln x.$

$\mu = e^{\int P(x)\,dx} = e^{\ln \frac{1}{x}} = \frac{1}{x}.$

$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 1.$

$\frac{d}{dx}\left(\frac{1}{x}y\right) = 1.$

$\frac{1}{x}y = x + C.$

$y = x^2 + Cx.$

Check Answer!

$\frac{dy}{dx} = 2x + C.$

$2x^2 + Cx - (x^2 + Cx) = x^2$ ✓

$y(1) = 1 \Rightarrow C = 0.$

p666 # 4

(a) the radius of convergence is 1

(b) Yes by the alternating ~~[illegible]~~ series test

2 (c) No $\sum \frac{1}{\sqrt{k}}$ is a p-series for $p = 1/2$ — Diverges.

(d) $3 \leq x < 5$.

p667 #39 $\sum_{k=0}^{\infty} \frac{100^k}{k!} x^k$.

$$\frac{|a_{k+1}|}{|a_k|} = \frac{100}{k+1}|x| \to 0 \quad \forall x$$

So $\sum \frac{100^k}{k!} x^k$ converges absolutely $\forall x$, so radius of convergence $= +\infty$, and the interval of convergence is $(-\infty, \infty)$

#46 $\sum_{k=1}^{\infty} \frac{(2k+1)!}{k^4} (x-5)^k$

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(2k+3)(2k+2)/(k+1)^4}{1/k^4} |x-5|$$

$$= \frac{(2k+3)(2k+2)\,k^4}{(k+1)^4} |x-5| \to \infty \quad \forall x \neq 5$$

So it converges only at $x=5$. radius of convergence is 0.

p 689.

(a) True – Divergence Test

(b) False $u_k = \frac{1}{k}$, $u_k \to 0$ but $\sum u_k$ diverges

(c) False $f(x) = \sin 2\pi x$.

$f(n) = 0 \quad \forall n$

So $f(n) \to 0$ but

$f(x) \not\to 0$ as $x \to \infty$

(d) True $f(x)$ ~~gets~~ arbitrarily close to $L \to f(n)$ get arbitrarily close to L.

(e) False $a_n = 1/4$ n even

$= 3/4$ n odd.

$0 < a_n < 1$ but a_n not converge

(f) False $u_k = \frac{1}{k}$.

(g) False $u_k = 1/k$.

(h) False $u_k = k$ $v_k = k^2$.

(i) True – comparison test

(j) True – " "

(k) False $\sum \frac{(-1)^k}{k}$ converges, not absolutely

(l) False $\sum \frac{(-1)^k}{k}$.

p 690 #16 (a) $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{2k+5}{k^2+k}$

$$\frac{2k+5}{k^2+k} = \frac{A}{k} + \frac{B}{k+1}$$

$$2k+5 = A(k+1) + Bk.$$

$$2k = (A+B)k.$$

$$5 = A \qquad \Rightarrow B = -3.$$

$$\frac{2k+5}{k^2+k} = \frac{5}{k} - \frac{3}{k+1}.$$

$k=1$ $\quad -5 + \frac{3}{2} = S_1$

$k=2$ $\quad \frac{5}{2} - \frac{3}{3} \quad S_2 = -5 + 8/2 - \frac{3}{3}$

$k=3$ $\quad -5/3 + \frac{3}{4}. \quad S_3 = -5 + 8/2 - 8/3 + 3/4$

$k=4$ $\quad 5/4 - 3/5. \quad S_4 = -5 + 8/2 - 8/3 + 8/4 - \frac{3}{5}$

$$S_n = -5 + 8\left(\frac{1}{2} - 1/3 + 1/4 - \pm \frac{1}{n}\right) \pm \frac{3}{n+1}$$

$$\downarrow \qquad \downarrow$$

$$-5 + 8 \sum_{n=2}^{\infty} (-1)^n \frac{1}{n} \pm 0.$$

Converges

p 690 #16 (c) $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{2k+5}{k^2+k}$

$$\frac{2k+5}{k^2+k} = \frac{A}{k} + \frac{B}{k+1}$$

$$2k+5 = A(k+1) + Bk$$

$$2k = (A+B)k$$

$$5 = A \quad \Rightarrow B = -3$$

$$\frac{2k+5}{k^2+k} = \frac{5}{k} - \frac{3}{k+1}$$

$k=1$ $\quad -5 + \frac{3}{2} = S_1$

$k=2$ $\quad \frac{5}{2} - \frac{3}{3} \quad S_2 = -5 + 8/2 - \frac{3}{3}$

$k=3$ $\quad -5/3 + \frac{3}{4} \quad S_3 = -5 + 8/2 - 8/3 + 3/4$

$k=4$ $\quad 5/4 - 3/5 \quad S_4 = -5 + 8/2 - 8/3 + 8/4 - \frac{3}{5}$

$$S_n = -5 + 8\left(\frac{1}{2} - 1/3 + 1/4 - \pm \frac{1}{n}\right) \pm \frac{3}{n+1}$$

$$\downarrow \qquad \downarrow$$

$$-5 + 8 \sum_{n=2}^{\infty} (-1)^n \frac{1}{n} \pm 0$$

Converges

(b) $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k+2}{3k-1}\right)^k \sim (-1)^k \left(\frac{1}{3}\right)^k$

Test for absolute convergence.

$$\sqrt[k]{\left|\frac{k+2}{3k-1}\right|^k} = \frac{k+2}{3k-1} \to \frac{1}{3}$$

So converges by the root test.

#18 $\sum_{k=1}^{\infty} \frac{\ln k}{k\sqrt{k}} \sim \frac{1}{k\sqrt{k}}$? since $\ln k \to \infty$ slowly

(b)

$$\lim_{k\to\infty} \frac{\ln k}{k^{1/4}} = \frac{\infty}{\infty} \stackrel{L}{=} \frac{\frac{1}{k}}{\frac{1}{4}k^{-3/4}} = \frac{4k^{3/4}}{k}$$

$$= \frac{4}{k^{1/4}} \to 0 \text{ as } k \to \infty$$

Hence $\frac{\ln k}{k^{1/4}} < 1$ eventually.

Hence $\frac{\ln k}{k\sqrt{k}} < \frac{1}{k \cdot k^{1/4}}$ − p series

$p = 1 + 1/4$

converges.

So $\sum \frac{\ln k}{k\sqrt{k}}$ converges

(b) $\sum \frac{k^{5/3}}{8k^2+5k+1} \sim \sum \frac{1}{k^{1/3}}$.

Use limit comparison test.

$$\frac{\dfrac{k^{5/3}}{8k^2+5k+1}}{\dfrac{1}{k^{1/3}}} = \frac{k^2}{8k^2+5k+1} \to \frac{1}{8}.$$

$\sum \frac{1}{k^{1/3}}$ p series $p = 1/3$ diverges

$\therefore \sum \frac{k^{5/3}}{8k^2+5k+1}$ diverges.

#26 $\sum \frac{(x-x_0)^k}{b^k}$

Ratio Test $\frac{|a_{k+1}|}{|a_k|} = \frac{|x-x_0|}{b} \to \frac{|x-x_0|}{b} < 1$

converges absolutely if $|x-x_0| < b$.

$x - x_0 = b$ $\quad \sum +1$ diverges

$x - x_0 = -b$ $\quad \sum (-1)^k$ diverges.

So interval of convergence is

$(x_0 - b, x_0 + b)$.