1 How would you integrate

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx \,?$$

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$$

Let $u = x+\frac{b}{2a}$

$$\frac{du}{dx} = 1.$$

$$= a\left(u^2+\frac{4ac-b^2}{4a^2}\right)$$

If $a>0$ and $b^2-4ac \leq 0$

$$4ac-b^2 \geqslant 0$$

$$= (\sqrt{a}\,u)^2+\left(\sqrt{\frac{4ac-b^2}{4a}}\right)^2$$

$$\int \frac{1}{\sqrt{x^2+d^2}}\,dx = \ln\left(x+\sqrt{x^2+d^2}\right)+c$$

see tables.

$a<0$ and $b^2-4ac \leq 0$

then ax^2+bx+c is always $\leqslant 0$ so $\sqrt{}$ does not exist except at best at a point

If $a > 0$ and $b^2 - 4ac \geq 0$

$4ac - b^2 \leq 0$

then $= (\sqrt{a}\, u)^2 - \left(\sqrt{\frac{|4ac-b^2|}{4a}}\right)^2$

$$\int \frac{1}{\sqrt{x^2 - d^2}}\,dx \longrightarrow \ln|x + \sqrt{x^2 - d^2}| + C$$

Note $\frac{1}{\sqrt{x^2 - d^2}}$ only exists

where $x^2 \geq d^2$

i.e. $|x| \geq |d|$

If $a < 0$ and $b^2 - 4ac \geq 0$.

i.e. $4ac - b^2 \leq 0$.

$= -(\sqrt{|a|} \cdot u)^2 + \left(\sqrt{\frac{4ac-b^2}{4a}}\right)^2$

$$\int \frac{1}{\sqrt{d^2 - x^2}}\,dx = \sin^{-1}\frac{x}{d} + C$$

Note $\frac{1}{\sqrt{d^2 - x^2}}$ only exists

where $d^2 \geq x^2$

i.e. $|d| \geq |x|$.

p576 #32

Polonium 210 has a ½ life of 140 days

20 mg in a lead container

$y(t)$ = # of milligrams after t days

(a) $y(140) = \frac{1}{2} y(0)$

$y(t) = y(0)\, e^{kt}$ $\qquad y(0) = 20.$

$\frac{1}{2} y(0) = y(0)\, e^{k140}$

$\frac{1}{2} = e^{k140}$

$\ln \frac{1}{2} = 140k$

$k = \dfrac{\ln \frac{1}{2}}{140}$

$= -.005.$

(b) $y(t) = 20\, e^{-.005t}$

(c) $y(70) = 20\, e^{-.005(70)}$

$= 14.01$ mg.

(d) $y(t_0) = \frac{3}{10} y(0) = 20\, e^{-.005t_0}$

$\frac{3}{10} = e^{-.005t_0}$

$\ln .3 = -.005t_0$

$t_0 = \ln .3 / -.005 = \cancel{210.94 \text{ days}}$ $\quad 240.79$ days

34. 10,000 in 1998.

12,000 in 2003

$$y(t) = y(0)\, e^{kt}$$ t = # of years after 1998

$y(0) = 10,000.$

$$y(5) = 10,000\, e^{k5} = 12,000$$

$$e^{5k} = \frac{6}{5}$$

$$5k = \ln 1.2.$$

$$k = \frac{\ln 1.2}{5}$$

$$k = .0365.$$

$$20,000 = 10,000\, e^{.0365 t_0}.$$

$$2 = e^{.0365 t_0}$$

$$\ln 2 = .0365 t_0.$$

$$t_0 = \frac{\ln 2}{.0365} = 18.990.$$

p5 94/5 18 3000 bacteria
growth 1% per hour

$y(t)$ = # of bacteria after t-hours

$\frac{dy}{dt} = .01y$. $y(0) = 3,000$.

$y(t) = 3,000\, e^{.01t}$

$y(t_0) = 6,000 = 3,000\, e^{.01t_0}$

$2 = e^{.01t_0}$

$\ln 2 = .01t_0$

$t_0 = \frac{\ln 2}{.01} = 69.315$ hrs

$y(t_0) = 30,000 = 3,000\, e^{.01t_0}$

$10 = e^{.01t_0}$

$t_0 = \frac{\ln 10}{.01} = 230.259$ hrs.

22.

$x\frac{dy}{dx} + 2y = 4x^2$ $y(1) = 3$.

Linear first order

Standard form $\frac{dy}{dx} + \frac{2}{x}y = 4x$

$P(x) = \frac{2}{x}$, $\rho(x) = e^{\int P(x)dx}$.

$= e^{2\ln x} = x^2$

$$x^2 \frac{dy}{dx} + 2xy = 4x^3.$$

$$\frac{d(x^2y)}{dx} = 4x^3.$$

$$\int dx \quad x^2 y = x^4 + c$$

$$y = x^2 + \frac{c}{x^2}$$

$x = 1$ $\quad 3 = 1 + c$

$$2 = c.$$

$$y = x^2 + \frac{2}{x^2}$$

28 $\quad y(t) =$ % of carbon monoxide in room

$$\frac{dy}{dt} = \text{\% carbon mon in} - \text{\% carbon mon. out}$$

$$= \frac{.1}{1{,}200} \times 4 - \frac{.1}{1{,}200} y.$$

$$\frac{dy}{dt} + \cancel{.0000833}\ \frac{.1}{1{,}200} y = \frac{.4}{1{,}200}$$

$$e^{\frac{.1}{1200}x} \frac{dy}{dt} + \frac{.1}{1200} e^{\frac{.1}{1200}x} y = \frac{.4}{1{,}200} e^{\frac{.1}{1200}x}.$$

$$\frac{d}{dt}\left(e^{\frac{.1}{1200}x} y\right) = \frac{.4}{1{,}200} \cdot e^{\frac{.1}{1200}x}.$$

$$e^{\frac{.1}{1200}x} y = 4 e^{\frac{.1}{1200}x} + c$$

$$y = 4 + c e^{-\frac{.1}{1200}x}.$$

$y(0) = 0$ so $c = -4$.

$$y(t) = 4 - 4e^{-\frac{.1}{1200}t}$$

$y(t_0) = .012\ \%$

$$= 4 - 4e^{-\frac{.1}{1200}\cdot(\text{[illegible]})}$$

$$.012 = 4 - 4e^{-\frac{.1}{1200}t_0}$$

$$3.988 = 4e^{-\frac{.1}{1200}t_0}$$

$$-\frac{.1}{1200}t_0 = \ln\frac{3.988}{4} = \ln .997$$

$t_0 = 36.054$ ~~hours~~ minutes.