April 2019.

1. (a) Complete and prove the following

    i. $(P \wedge Q)' \equiv$

    ii. $(P \to Q)' \equiv$

   (b) Is the following a valid argument? $P \wedge Q$, and Q', therefore P'. Be sure to prove your answer.

   (c) State Zorn's Lemma and use it to prove that every vector space has a basis.

2. (a) Define denumerable set. If each set A_i is countable, prove or disprove

    i. $\bigcup_{i=1,\infty} A_i$ is denumerable.

    ii. $\prod_{i=1,\infty} A_i$ is denumerable.

   (b) State the Schroeder-Bernstein Theorem and Cantor's Theorem. Prove Schroeder-Bernstein.

   (c) If f maps X to Y define f^{-1} from the power set of Y to the power set of X, and Prove $f^{-1}(\bigcap_{i=1,\infty} A_i) = \bigcap_{i=1,\infty} f^{-1}(A_i)$.

3. (a) Define open set in $\mathbb{R}$, and closed set. Prove that the union of any collection of open sets is open. Show by example that the intersection of a collection of open sets need not be open.

   (b) Define Cauchy sequence. Prove that the property of every Cauchy sequence of real numbers having a real limit is equivalent to the least upper bound axiom.

4. (a) Define connected subset of $\mathbb{R}$. Prove a subset A is connected only if it is an interval.

   (b) Prove that the continuous image of a connected subset of $\mathbb{R}$ is connected.

   (c) Use part (b) to prove the Intermediate Value Theorem.

Solutions MA1126 April 2019.

1 (a) (i) $(P \wedge Q)' \equiv P' \vee Q'$ $\quad$ ' means NOT

P	Q	$P \wedge Q$	$(P \wedge Q)'$	P'	Q'	$P' \vee Q'$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T
			*			*

$* = *$ so $(P \wedge Q)' \equiv P' \vee Q'$

(ii) $(P \rightarrow Q)' \equiv P \wedge Q'$

P	Q	$P \rightarrow Q$	$(P \rightarrow Q)'$	Q'	$P \wedge Q'$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F
			*		*

$* = *$ so $(P \rightarrow Q)' \equiv P \wedge Q'$

(b) $(P \wedge Q) \wedge Q' \Rightarrow P'$

P	Q	$P \wedge Q$	Q'	$(P \wedge Q) \wedge Q'$	P'	$(P \wedge Q) \wedge Q' \Rightarrow P'$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	F	F	F	T	T
F	F	F	T	F	T	T
						*

$* \Rightarrow$ Tautology $\Rightarrow$ Valid argument.

1 (c) Zorn's Lemma: If $(X, \leq)$ is a partially ordered set such that every totally ordered subset has an upper bound then X has a maximal element.

Thm: Every vector space has a basis.

Note a basis is a minimal spanning set or a maximal linearly independent set for V

We use the latter

Let $X = \{$ all lin. indept. subsets of $V\}$. If U_1 and $U_2 \in X$, define $U_1 \leq U_2$ means $U_1 \subset U_2$.

Let $\{U_\alpha\}$ be a totally ordered subset of X.

Let $U = \bigcup_\alpha U_\alpha$. Clearly $U_\alpha \leq U$ $\forall \alpha$,

but we need $U \in X$, ie U is a lin. indept. subset of V.

Let $x_1, \ldots, x_n \in U$ and $\sum_{i=1}^{n} \beta_i x_i = 0$

β_i scalars.

$x_i \in \bigcup_\alpha U_\alpha \Rightarrow \forall i \; \exists \, U_{\alpha_i}, \; x_i \in U_{\alpha_i}$

Now $U_{\alpha_1} \subset U_{\alpha_2}$ or $U_{\alpha_2} \subset U_{\alpha_1}$

ie $x_1, x_2 \in U_{\alpha_2}$ or $x_1, x_2 \in U_{\alpha_1}$.

$x_1 \ldots x_n \in$ some $U_{\alpha_{i_0}} \Rightarrow$ lin. indept.

$\therefore$ By Zorn $\exists$ a basis.

2 (a) A set X is denumerable if $\exists f$, 1-1 and onto $f: X \to \mathbb{N}$. i.e. X can be listed $x_1, x_2, x_3, \dots, x_n, \dots$

(i) $\bigcup_{i=1,\infty} A_i$ is denumerable.

$$
\begin{aligned}
A_1 &= x_{11} \to x_{12} \quad x_{13} \quad \cdots \quad x_{1n} \cdots \\
A_2 &= x_{21} \quad x_{22} \quad x_{23} \quad \cdots \quad x_{2n} \\
A_3 &= x_{31} \quad x_{32} \quad x_{33} \\
&\vdots \\
A_m &= x_{m1} \quad x_{m2} \quad x_{m3} \quad \cdots \quad x_{mn} \\
&\vdots
\end{aligned}
$$

Now follow the arrows to list $\bigcup A_i$

(ii) $\prod_{i=1,\infty} A_i$ is not necessarily denumerable

$\prod \{0,1\}$ = all sequences of 0's and 1's

Claim $\prod\{0,1\} \sim \mathcal{P}(\mathbb{N})$.

Let $f: \mathcal{P}(\mathbb{N}) \to \prod\{0,1\}$
by $(f(A))_i = 1$ if $i \in A$
$= 0$ if $i \notin A$.

Easy to check f is 1-1 and onto.

But $\# \mathcal{P}(\mathbb{N}) > \# \mathbb{N}$ by Cantor's Thm

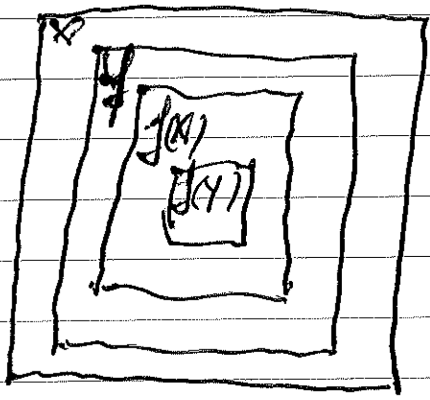
So $\mathcal{P}(\mathbb{N})$ is not denumerable.

2 (b) Schroeder-Bernstein. If $\exists f$ 1-1 $X \to Y$ and $\exists g$ 1-1 $Y \to X$, then $\exists h: X \to Y$ which is 1-1 and onto.

or $X \le Y$ and $Y \le X \Rightarrow X \sim Y$

Cantor's Thm: $\# \mathcal{P}(X) > \# X$.

Pf of S-B We can use g to identify Y with a subset of X. Consider



etc.

Now $X = X \setminus Y \cup (Y \setminus f(X)) \cup (f(X) \setminus f(Y)) \cup f(Y) \setminus f^2(X)) \cup (f^2(X) \setminus f^2(Y)) \cup \ldots \cup \bigcap f^n(X)$

and $Y = (Y \setminus f(X)) \cup (f(X) \setminus f(Y)) \cup (f(Y) \setminus f^2(X)) \cup (f^2(X) \setminus f^2(Y)) \cup \ldots \cup \bigcap f^n(X)$

Let $h: X \setminus Y \to f(X) \setminus f(Y)$ via f.

$h: Y \setminus f(X) \to Y \setminus f(X)$ via id.

$\vdots$

$\bigcap f^n(X) \to \bigcap f^n(X)$ via id.

By construction h is onto. Is h 1-1?

Lemma If h is 1-1 on A and h is 1-1 on B and $h(A) \cap h(B) = \phi$, then h is 1-1 on $A \cup B$.

Pf: Let $h(x_1) = h(x_2)$ $x_1, x_2 \in A \cup B$.

Case 1. $x_1, x_2 \in A$, $\Rightarrow x_1 = x_2$ by h 1-1 on A

2. $x_1, x_2 \in B$, $\Rightarrow x_1 = x_2$ by h 1-1 on B

3. $x_1 \in A$, $x_2 \in B$ or vice versa impossible since $h(A) \cap h(B) = \phi$.

By the Lemma h is 1-1 on X, and we are done.

(c) $f: X \to Y$, let $A \subset Y$, define

$f^{-1}(A) = \{x : f(x) \in A\}$, then $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$

Let $A_i \in \mathcal{P}(Y)$.

$$x \in f^{-1}(\cap A_i) \Leftrightarrow f(x) \in \cap A_i$$

$$\Leftrightarrow f(x) \in A_i \quad \forall i.$$

$$\Leftrightarrow x \in f^{-1}(A_i) \quad \forall i$$

$$\Leftrightarrow x \in \cap f^{-1}(A_i)$$

3 (a) O is open in $\mathbb{R}$ if every point in O is an interiore point ie $\forall x \in O, \exists \epsilon$ s.t. $N_\epsilon(x) \subset O$.

G is closed means G^c is open

Let $A = \bigcup_\alpha O_\alpha$. Let $x \in A$, then $\exists \alpha_0$ s.t. $x \in O_{\alpha_0}$, open $\Rightarrow \exists \epsilon$ s.t. $N_\epsilon(x) \subset O_{\alpha_0}$ $\Rightarrow N_\epsilon(x) \subset A$ $\therefore$ A is open

Let $O_n = (-\frac{1}{n}, \frac{1}{n})$ then $\bigcap O_n = \{0\}$ and 0 is not an interion point so $\{0\}$ is not open.

(b) $\{x_n\}$ is a Cauchy sequence means $\forall \epsilon > 0$ $\exists N$ s.t. $n, m \geq N$ $\Rightarrow |x_n - x_m| < \epsilon$.

Next C.S. converge $\Leftrightarrow$ l.u.b. axiom.

We know l.u.b. axiom $\Rightarrow$ Nested Int. Prop
$\Rightarrow$ Every infinite bounded subset of $\mathbb{R}$ has an acc. pt.

Let $\{x_n\}$ be a C.S in $\mathbb{R}$. Let $\epsilon = 1$, $\exists N$ s.t $n, m \geq N \Rightarrow |x_n - x_m| < 1$.

Let $M = \max\{x_1, x_2, \ldots, x_{N-1}, x_N + 1\}$
then $x_n \leq M$ $\forall n$.

$m = \min\{x_1, x_2, \ldots x_{N-1}, x_N - 1\}$
then $x_n \geq m$ $\forall n$

So C.S. $\Rightarrow$ bounded set.

3. If x_n are not infinitely distinct then we have a subsequence $\to x$ say & if infinitely distinct, then by Bolzano Weierstrass we have a convergent subsequence. Say $\{x_{i_n}\} \to x$.

Then $\forall \epsilon \; \exists \; i_{n_0}$ s.t $i_n \geq i_{n_0}$
$\Rightarrow |x_{i_n} - x| < \epsilon/2$

& $\exists \; N_1$ s.t. $n, m > N_1$
$\Rightarrow |x_n - x_m| < \epsilon/2$.

Choose N s.t $N \geq i_{n_0}$ & $N > N_1$

then $|x_n - x| \leq |x_n - x_{i_{n_0}}| + |x_{i_{n_0}} - x|$
$\leq \epsilon$ if $n \geq N$

$\therefore \; x_n \to x$.

Conversely if every C.S. in $\mathbb{R}$ converges we want to show that $\mathbb{R}$ has the l.u.b. property.

Let A be a non-empty set bounded above by b_0 say.
$\exists \; a_0 \in A$, $a_0 \leq b_0$

Consider the mid point $\frac{b-a}{2}$
Either $\frac{b-a}{2}$ is an upper bound
if so let $b_1 = \frac{b-a}{2}$, $a_1 = a_0$

3.

as $\frac{b-a}{2}$ is not an upper bound, $\exists\ a_1 \in A$

$a_1 \geq \frac{b-a}{2}$, let $b_1 = b_0$.

Continue taking mid points.

We get a sequence $a_n \leq b_n$.

$d(a_n, b_n) \to 0$

$\Rightarrow$ $\{a_n\}$ + $\{b_n\}$ are C.S.

$\Rightarrow$ $a_n \to x$ + $b_n \to x$

+ x is the l.u.b. Check

b_n are upper bounds $\Rightarrow$ x is an upper bound and $a_n \in A$ $a_n \to x$

$\Rightarrow$ no smaller upper bound.

4 (a) $A \subset \mathbb{R}$ is disconnected means $\exists\ O_1, O_2$ open such that

1. $A \subset O_1 \cup O_2$
2. $A \cap O_i \neq \emptyset$ $i = 1, 2$
3. $O_1 \cap O_2 = \emptyset$

A is connected means A is not disconnected.

4 (a) If A is not an interval, then $\exists$ $x, y \in A$, and $x < z < y$ with $z \notin A$.

Let $O_1 = (-\infty, z)$ and $O_2 = (z, +\infty)$

then $A \subset O_1 \cup O_2$, $O_1 \cap A \supset \{x\}$

$O_2 \cap A \supset \{y\}$ + $O_1 \cap O_2 = \emptyset$

$\therefore$ A is disconnected.

(b) f continuous, A connected.

Suppose $f(A)$ is disconnected.

$\Rightarrow$ $f(A) \subset O_1 \cup O_2$ $\quad O_i \cap f(A) \neq \emptyset$

$O_1 \cap O_2 = \emptyset$

$\Rightarrow$ $A \subset f^{-1}(O_1) \cup f^{-1}(O_2)$

f cont, O_i open $\Rightarrow$ $f^{-1}(O_i)$ open

$O_i \cap f(A) \neq \emptyset \Rightarrow f^{-1}(O_i) \cap A \neq \emptyset$.

+ $O_1 \cap O_2 = \emptyset \Rightarrow f^{-1}(O_1) \cap f^{-1}(O_2) = \emptyset$

$\therefore$ A is disconnected.

This is a contradiction, so $f(A)$ is connected.

4 (c) The Intermediate Value Theorem says that if f is continuous on $[a, b]$ and $f(a) < d < f(b)$ then $\exists\, c \in [a, b]$ s.t. $f(c) = d$.

<u>Pf</u>: $[a, b]$ is an interval so $[a, b]$ is connected. f cont $\Rightarrow f([a, b])$ is connected by (b) $\Rightarrow f([a, b])$ is an interval by (a) $\Rightarrow$ Theorem.