1. (a) Complete and prove the following
   
   i. \( (P \land Q)' \equiv \)
   
   ii. \( (P \rightarrow Q)' \equiv \)

   (b) Is the following a valid argument? \( P \land Q \), and \( Q' \), therefore \( P' \). Be sure to prove
   your answer.

   (c) State Zorn's Lemma and use it to prove that every vector space has a basis.

2. (a) Define denumerable set. If each set \( A_i \) is countable, prove or disprove
   
   i. \( \bigcup_{i=1,\infty} A_i \) is denumerable.
   
   ii. \( \prod_{i=1,\infty} A_i \) is denumerable.

   (b) State the Schroeder-Bernstein Theorem and Cantor's Theorem. Prove Schroeder-
   Bernstein.

   (c) If \( f \) maps \( X \) to \( Y \) define \( f^{-1} \) from the power set of \( Y \) to the power set of \( X \), and
   Prove \( f^{-1}(\bigcap_{i=1,\infty} A_i) = \bigcap_{i=1,\infty} f^{-1}(A_i) \).

3. (a) Define open set in \( \mathbb{R} \), and closed set. Prove that the union of any collection of
   open sets is open. Show by example that the intersection of a collection of open
   sets need not be open.

   (b) Define Cauchy sequence. Prove that the property of every Cauchy sequence of
   real numbers having a real limit is equivalent to the least upper bound axiom.

4. (a) Define connected subset of \( \mathbb{R} \). Prove a subset \( A \) is connected only if it is an
   interval.

   (b) Prove that the continuous image of a connected subset of \( \mathbb{R} \) is connected.

   (c) Use part (b) to prove the Intermediate Value Theorem.
Solutions MA1126 April 2019.

1 (a) i) \((P \land Q)' \equiv P' \lor Q'\)  
   means NOT

\[
\begin{array}{ccccccc}
P & Q & P \land Q & (P \land Q)' & P' & Q' & P' \lor Q' \\
T & T & T & F & F & F & F \\
T & F & F & T & F & T & T \\
F & T & F & T & T & F & T \\
F & F & F & T & T & T & T \\
\end{array}
\]

\(\times \times \times \times\)

\(\times = \times\)  \(\equiv (P \land Q)' \equiv P' \lor Q'\)

(ii) \((P \rightarrow Q)' \equiv P \land Q'\)

\[
\begin{array}{ccccccc}
P & Q & P \rightarrow Q & (P \rightarrow Q)' & Q' & P \land Q' \\
T & T & T & F & F & F \\
T & F & F & T & T & T \\
F & T & T & F & F & F \\
F & F & T & T & F & F \\
\end{array}
\]

\(\times \times \times \times\)

\(\times = \times\)  \(\equiv (P \rightarrow Q)' \equiv P \land Q'\)

(b) \((P \land Q) \land Q' \equiv P'\)

\[
\begin{array}{ccccccc}
P & Q & P \land Q & (P \land Q)' & Q' & (P \land Q) \land Q' & P' \\
T & T & T & F & F & F & T \\
T & F & F & T & T & F & T \\
F & T & F & F & T & F & T \\
F & F & F & T & T & T & T \\
\end{array}
\]

\(\times = \text{TAUTOLOGY} \Rightarrow \text{VALID ARGUMENT}\)
Zorn's Lemma: If (X, ≤) is a partially ordered set such that every totally ordered subset has an upper bound, then X has a maximal element.

Thm: Every vector space has a basis.

Note a basis is a minimal spanning set or a maximal linearly independent set for V.

We use the latter.

Let \( X = \{ \text{all lin. indept. subsets of } V \} \).

Let \( U_1 \) and \( U_2 \in X \), define \( U_1 \leq U_2 \) means \( U_1 \subseteq U_2 \).

Let \( U_0 \) be a totally ordered subset.

Let \( U = U_0 \cap X \). Clearly \( U \subseteq V \).

But we need \( U \in X \), i.e., \( U \) is a lin. indept. subset of \( V \).

Let \( \alpha_1, \ldots, \alpha_n \in U \) and \( \sum_{i=1}^{n} \beta_i \alpha_i = 0 \) scalars.

\[ \alpha_i \in U_0 \implies \forall \alpha_i \in U_0, \alpha_i \in U_0 \]

Hence \( U_0 \subseteq U_0 \) or \( U_0 \subseteq U_0 \).

E.g. \( x_1, x_2 \in U_0 \implies \alpha_1, \alpha_2 \in U_0 \).

\( \therefore x_1, x_2 \in U_0 \implies \text{lin. indept.} \).

By Zorn \( \exists \subseteq \text{ basis.} \)
2. A set $X$ is denumerable if $\exists f, g: 1 \rightarrow 1$ and onto $\exists f: X \rightarrow N$. In $X$ can be
denoted by $x_1, x_2, x_3, \ldots, x_n, \ldots$

$\exists f, g, n$

$U A_n$ is denumerable.

$A_1 = x_1, x_2, x_3, \ldots, x_n, \ldots$

$A_2 = x_2, x_2, x_2, \ldots, x_2, \ldots$

$A_3 = x_3, x_3, x_3, \ldots, x_3, \ldots$

$\vdash A_n = x_{n1}, x_{n2}, x_{n3}, \ldots, x_{nn}$

Now follow the arrows to find $U A_n$.

(ii) $\Pi A \subset X$ is not necessarily denumerable.

$\Pi \{0, 1\} = \{\text{all sequences of 0 and 1}\}$

Claim: $\Pi \{0, 1\} \subset 2^N$.

Let $f: 2^N \rightarrow \Pi \{0, 1\}$

$s.t. (f(A))_i = 1$ if $i \in A$

$= 0$ if $i \notin A$.

Easy to check $f$ is 1-1 and onto.

But $\# 2^N > \# N$ by Cantor’s Theorem.

$\exists (2^N)$ is not denumerable.
2. (b) Schroeder-Bernstein. If \( f: X \rightarrow Y \) and \( g: Y \rightarrow X \) then \( f \circ g: X \rightarrow X \) which is 1-1 and onto.

or \( X \leq Y \) and \( Y \leq X \) \( \Rightarrow X \sim Y \)

Cantor's Theorem: \( \# P(X) > \# X \).

Proof of S-B: We can use \( g \) to identify \( Y \) with a subset of \( X \). Consider

\[ X = X \backslash \{ y_1 \} \cup (y_1 \backslash g^{-1}(x_1)) \cup (g^{-1}(x_1) \backslash y_1) \cup g^{-1}(x_1) \cup y_1 \backslash g^{-1}(x_1) \cup \cdots \]

and

\[ Y = (X \backslash f(x_1)) \cup (f(x_1) \backslash y_1) \cup (y_1 \backslash f(x_1)) \cup f(x_1) \cup x_1 \backslash f(x_1) \cup \cdots \]

Let \( h: X \sim Y \) \( \Rightarrow f \) and \( g(h(x)) \). Then \( f \backslash (\text{id}) \) and \( g \).

\[ f(h(x)) \sim (x \backslash y_1) \cup y_1 \backslash (x \backslash y_1) \sim \text{id}. \]
By construction \( h \) is onto. Is \( h \) 1-1?

**Lemma.** If \( h \) is 1-1 on \( A \) and \( h \) is 1-1 on \( B \) and \( h(A) \cap h(B) = \emptyset \), then \( h \) is 1-1 on \( A \cup B \).

**Proof.** Let \( h(x_1) = h(x_2) \) \( x_1, x_2 \in A \cup B \).

1. Case 1. \( x_1, x_2 \in A \) \( \Rightarrow x_1 = x_2 \) by \( h \) 1-1 on \( A \).
2. Case 2. \( x_1, x_2 \in B \) \( \Rightarrow x_1 = x_2 \) by \( h \) 1-1 on \( B \).
3. Case 3. \( x_1 \in A, x_2 \in B \) or vice versa.

Impossible since \( f(A) \cap f(B) = \emptyset \).

By the lemma, \( h \) is 1-1 on \( X \), and we are done.

(a) \( f : X \to Y \), let \( A \subseteq Y \), define

\[ f^{-1}(A) = \{ x : f(x) \in A \} \]

Then \( f^{-1}(A) = \bigcup_{A_i \in \mathcal{P}(Y)} f^{-1}(A_i) \).

\[ x \in f^{-1}(A) \iff f(x) \in A \iff f(x) \in A_i \iff x \in f^{-1}(A_i) \iff x \in \bigcup_{A_i \in \mathcal{P}(Y)} f^{-1}(A_i) \]
3. (a) $O$ is open in $\mathbb{R}$ if every point in $O$ is an interior point i.e. $x \in O, \exists \epsilon > 0$ s.t. $N_{x, \epsilon} \subset O$.

(b) $G$ is closed means $G^c$ is open.

Let $A = \bigcup O_{x, h}$. Let $x \in A$, then $\exists \epsilon$ s.t. $x \in O_{x, \epsilon}$, open $\implies \exists \delta > 0$ s.t. $N_{x, \delta} \subset O_{x, \epsilon}$

$\implies x \in O_{x, \delta}$, i.e., $A$ is open.

Let $O_n = (\frac{1}{n}, \frac{1}{n})$ then $\cap O_n = \emptyset$

and $O$ is not an interior point $\implies \cap$ is not open.

(b) $\{x_n\}$ is a Cauchy sequence means

$\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. $x_n, x_m \in N$

$\implies |x_n - x_m| < \epsilon$.

Next C.S. converge $\iff$ l.u.b. axiom.

We know l.u.b. axiom $\implies$ Nested Int. Prop.

$\implies$ Every infinite bounded subset of $\mathbb{R}$ has an acc. pt.

Let $\{x_n\}$ be a C.S. in $\mathbb{R}$, let $\epsilon > 0$

$s.t. n, m \geq N \implies |x_n - x_m| < \epsilon$.

Let $M = \max \{x_1, x_2, \ldots, x_{N-1}, x_N + \epsilon\}$ then $x_n \leq M \forall n$.

$m = \min \{x_1, x_2, \ldots, x_{N-1}, x_N - \epsilon\}$

then $x_n \geq m \forall n$.

So C.S. $\iff$ bounded set.
3. If \( x_n \) are not infinitely distinct, then we have a subsequence \( x_{s_n} \). If infinitely distinct, then by Bolzano-Weierstrass we have a convergent subsequence. Say \( x_{s_{n_k}} \to x \).

Then \( A \subseteq I \), s.t. \( x_{s_{n_k}} \to x \)

\[ |x_{s_{n_k}} - x| < \epsilon_1 \]

\[ \exists \ N_1 \text{ s.t. } n, m > N_1 \]

\[ |x_{s_{n_k}} - x_{s_{n_m}}| < \epsilon_2 \]

Choose \( N \) s.t. \( N > N_1 \)

\[ |x_n - x| \leq |x_n - x_{s_{n_k}}| + |x_{s_{n_k}} - x| \]

\[ < \epsilon \text{ if } n > N \]

\[ \therefore x_n \to x. \]

Conversely, if every C.S. in \( R \) converges, we want to show that \( R \) has the L.U.B. property.

Let \( A \) be a non-empty set bounded above by \( b_0 \). Say,

\[ \exists a \in A, \quad a \leq b_0 \]

Consider the mid point \( \frac{b-a}{2} \)

Either \( \frac{b-a}{2} \) is an upper bound

If no, let \( b_1 = \frac{b-a}{2}, \quad a_1 = a_0 \)
1. \( \frac{b-a}{2} \) is not an upper bound for \( B \).

\[
a_i < \frac{b-a}{2}, \quad \text{but } b_1 = b_0.
\]

Continue taking mid points.

We get a sequence \( a_n < b_n \).

\[
a \left( a_n, b_n \right) \rightarrow a
\]

\[
\Rightarrow \quad a_n + \frac{b_n}{2}, \quad \text{as } c, c
\]

\[
a_n \rightarrow a + b_n \rightarrow x
\]

\( x \) is the l. u. b. Check

\( b_n \) an upper bound \( \Rightarrow x \) is an upper bound and \( a_n \in A \) \( \Rightarrow x \)

is the smaller upper bound.

4. (b) A is connected means

\[
\exists \ O_1, O_2 \text{ open such that }
\]

1. \( A \subseteq O_1 \cup O_2 \)

2. \( A \cap O_i \neq \emptyset \quad i = 1, 2 \)

3. \( O_1 \cap O_2 = \emptyset \)

A is connected means \( A \) is not disconnected.
4 (a) If \( A \) is not an interval, then \( z \notin A \), and \( x < z < y \) with \( z \neq A \).

Let \( O_1 = (-\infty, z) \) and \( O_2 = (z, +\infty) \)

then \( A \subset O_1 \cup O_2 \), \( O_1 \cap A = \emptyset \)

\( O_2 \cap A = \emptyset \) \( \Rightarrow \ O_1 \cap O_2 = \emptyset \)

\( \therefore \ A \) is disconnected.

(b) If continuous, \( A \) connected.

Suppose \( f(A) \) is disconnected.

\[ O \]
\[ f(A) \subset O_1 \cup O_2 \quad O_1 \cap O_2 = \emptyset \]

\( = \quad A \subset f^{-1}(O_1) \cup f^{-1}(O_2) \)

\( \text{If cont., } O \text{ open } \Rightarrow f^{-1}(O) \text{ open} \)

\( f^{-1}(O_1) \neq \emptyset \Rightarrow f^{-1}(O_1) \cap A \neq \emptyset \)

\( + \quad O_2 \cap O_2 = \emptyset \Rightarrow f^{-1}(O_1) 
\)

\( \therefore \ A \) is disconnected.

This is a contradiction, so \( f(A) \)

is connected.
4. (c) The Intermediate Value Theorem says that if \( f \) is continuous on \( \mathbb{R} \) and \( f(a) < d < f(b) \) then there exists \( c \in (a, b) \) such that \( f(c) = d \).

Proof: \( [a, b] \) is an interval on \( \mathbb{R} \), it is connected. \( f \) is continuous on \( [a, b] \), it is connected by (b). \( \Rightarrow \) \( f([a, b]) \) is an interval by (a) \( \Rightarrow \) Theorem.