Assignment 5 MA1126.

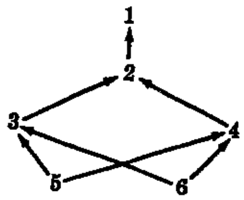
Q1. Use Zorn's Lemma to prove, any A, B.

$A < B$, $A \sim B$ or $B < A$.

From p 46 – copied here

#40, 43, 46.

37. Let $X = \{1, 2, 3, 4, 5, 6\}$ be ordered as in the adjacent diagram. Consider the subset $A = \{2, 3, 4\}$ of X. (i) Find the maximal elements of X. (ii) Find the minimal elements of X. (iii) Does X have a first element? (iv) Does X have a last element? (v) Find the set of upper bounds of A. (vi) Find the set of lower bounds of A. (vii) Does $\sup(A)$ exist? (viii) Does $\inf(A)$ exist?



38. Consider Q, the set of rational numbers, with the natural order, and its subset $A = \{x : x \in Q, x^3 < 3\}$. (i) Is A bounded above? (ii) Is A bounded below? (iii) Does $\sup(A)$ exist? (iv) Does $\inf(A)$ exist?

39. Let **N**, the positive integers, be ordered by "x divides y", and let $A \subset \mathbf{N}$. (i) Does $\inf(A)$ exist? (ii) Does $\sup(A)$ exist?

40. Prove: Every finite partially ordered set has a maximal element.

41. Give an example of an ordered set which has exactly one maximal element but does not have a last element.

42. Prove: If R is a partial order on A, then R^{-1} is also a partial order on A.

ZORN'S LEMMA

43. Consider the proof of the following statement: There exists a finite set of positive integers which is not a proper subset of any other finite set of positive integers.

Proof. Let $\mathcal{A}$ be the class of all finite sets of positive integers. Partially order $\mathcal{A}$ by set inclusion. Now let $\mathcal{B} = \{B_i : i \in I\}$ be a totally ordered subclass of $\mathcal{A}$. Consider the set $A = \cup_i B_i$. Observe that $B_i \subset A$ for every $B_i \in \mathcal{B}$; hence A is an upper bound of $\mathcal{B}$.

Since every totally ordered subset of $\mathcal{A}$ has an upper bound, by Zorn's Lemma, $\mathcal{A}$ has a maximal element, a finite set which is not a proper subset of another finite set.

Question: Since the statement is clearly false, which step in the proof is incorrect?

44. Prove the following fact which was assumed in the proof in Problem 24: Let $\{f_i : A_i \to B\}$ be a class of functions which is totally ordered by set inclusion. Then $\cup_i f_i$ is a function from $\cup_i A_i$ into B.

45. Prove that the following two statements are equivalent:

(i) (Axiom of Choice.) The product $\prod\{A_i : i \in I\}$ of a non-empty class of non-empty sets is non-empty.

(ii) If $\mathcal{A}$ is a non-empty class of non-empty disjoint sets, then there exists a subset $B \subset \bigcup\{A : A \in \mathcal{A}\}$ such that the intersection of B and each set $A \in \mathcal{A}$ consists of exactly one element.

46. Prove: If every totally ordered subset of an ordered set X has a lower bound in X, then X has a minimal element.