37. Let \( X = \{1, 2, 3, 4, 5, 6\} \) be ordered as in the adjacent diagram. Consider the subset \( A = \{2, 4, 5\} \) of \( X \). (i) Find the maximal elements of \( X \). (ii) Find the minimal elements of \( X \). (iii) Does \( X \) have a first element? (iv) Does \( X \) have a last element? (v) Find the set of upper bounds of \( A \). (vi) Find the set of lower bounds of \( A \). (vii) Does \( \sup(A) \) exist? (viii) Does \( \inf(A) \) exist?

38. Consider \( Q \), the set of rational numbers, with the natural order, and its subset \( A = \{x : x \in Q, x^3 < 3\} \). (i) Is \( A \) bounded above? (ii) Is \( A \) bounded below? (iii) Does \( \sup(A) \) exist? (iv) Does \( \inf(A) \) exist?

39. Let \( N \), the positive integers, be ordered by "\( z \) divides \( y \)" and let \( A \subset N \). (i) Does \( \inf(A) \) exist? (ii) Does \( \sup(A) \) exist?

40. Prove: Every finite partially ordered set has a maximal element.

41. Give an example of an ordered set which has exactly one maximal element but does not have a last element.

42. Prove: If \( B \) is a partial order on \( A \), then \( B^{-1} \) is also a partial order on \( A \).

**ZORN'S LEMMA**

43. Consider the proof of the following statement: There exists a finite set of positive integers which is not a proper subset of any other finite set of positive integers.

**Proof:** Let \( \mathcal{A} \) be the class of all finite sets of positive integers. Partially order \( \mathcal{A} \) by set inclusion. Now let \( \mathcal{B} = \{B_i : i \in I\} \) be a totally ordered subclass of \( \mathcal{A} \). Consider the set \( A = \bigcup B_i \). Observe that \( B_i \subseteq A \) for every \( B_i \in \mathcal{B} \); hence \( A \) is an upper bound of \( \mathcal{B} \).

Since every totally ordered subset of \( \mathcal{A} \) has an upper bound, by Zorn's Lemma, \( \mathcal{A} \) has a maximal element, a finite set which is not a proper subset of another finite set.

**Question:** Since the statement is clearly false, which step in the proof is incorrect?

44. Prove the following fact which was assumed in the proof of Problem 24: Let \( \{f_i : A_i \to B\} \) be a class of functions which is totally ordered by set inclusion. Then \( \bigcup f_i \) is a function from \( \bigcup A_i \) into \( B \).

45. Prove that the following two statements are equivalent:
   (i) (Axiom of Choice.) The product \( \prod \{A_i : i \in I\} \) of a non-empty class of non-empty sets is non-empty.
   (ii) If \( \mathcal{A} \) is a non-empty class of non-empty disjoint sets, then there exists a subset \( B \subseteq \bigcup \{A : A \in \mathcal{A}\} \) such that the intersection of \( B \) and each set \( A \in \mathcal{A} \) consists of exactly one element.

46. Prove: If every totally ordered subset of an ordered set \( X \) has a lower bound in \( X \), then \( X \) has a minimal element.