1. (a) Construct the true/false table for $(P \wedge Q') \rightarrow (Q \vee P)$ Is there a name for this sort of statement?

   (b) Complete and prove the following

      i. $A \cap (B \cup C) =$

      ii. $(A \cap B)^c =$

   (c) Consider the following statements

      i. If John does not wear a coat then he is warm

      ii. John is cold is sufficient for his wearing a coat

      iii. John is warm or he is wearing a coat

      iv. Wearing a coat is a necessary condition for John to be cold

   Which of the above are equivalent to If John is cold then he is wearing a coat.

   (d) Prove that the power set of a set with n elements has 2^n elements.

2. (a) Define what it means for two sets to have the same cardinal number. Define countable set. Prove that $\mathbb{Q}$ is countable but $\mathbb{R}$ is not.

   (b) Define partial order and total or linear order. State the Axiom of Choice. Define well ordered set. Prove that if every set can be well ordered then the Axiom of Choice holds.

   (c) State the Schroeder-Bernstein Theorem and Cantor's Theorem. Prove one of them.

3. (a) Prove or disprove O_n open $\rightarrow \cap_n O_n$ open, O_n open $\rightarrow \cup_n O_n$ open. Then same questions for closed sets.

   (b) Prove that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

4. (a) Show that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous iff for every open set $O, f^{-1}(O)$ is open.

   (b) Prove that the continuous image of a compact subset of $\mathbb{R}$ is compact.

   (c) Use part (b) to prove the Extreme Value Theorem.

Solutions MA1126 May 2017.

Q 1.

(a)

P	Q	Q'	$P\wedge Q'$	$Q \vee P$	$P\wedge Q' \to Q\vee P$
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	F	T

This sort of statement is called a tautology.

(b) (i) $A \cap (B\cup C) = (A\cap B)\cup(A\cap C)$

$A = \{x : P(x)\}$ $B = \{x : Q(x)\}$ $C = \{x : R(x)\}$

$$A\cap(B\cup C) = \{x : P(x)\wedge (Q(x)\vee R(x))\}$$
$$= \{x : (P(x)\wedge Q(x))\vee(P(x)\wedge R(x))\}$$
$$= \{x : P(x)\wedge Q(x)\}\cup\{x : P(x)\wedge R(x)\}$$
$$= (A\cap B)\cup(A\cap C).$$

(ii) $$(A\cap B)^c = \{x : (P(x)\wedge Q(x))'\}$$
$$= \{x : P(x)'\vee Q(x)'\}$$
$$= A^c \cup B^c.$$

(c) P: John is cold Q he is wearing a coat

Statement $P \to Q$.

(i) $Q' \to P'$ Contrapositive is equivalent.

(ii) $P \to Q$ equivalent

(iii) $P' \vee Q$ equivalent

(iv) $P \to Q$ equivalent.

(d) $\# X = 1 \Rightarrow \# P(X) = 2^1$ $P(X) = \{X, \emptyset\}$

By induction.

if $\# P(X) = 2^k$ for $\# X = k$.

if X has $k+1$ elements

$= \cancel{k}$ elts + 1 element

X_1 + 1 elt.

$\# P(X_1) = 2^k$

every subset of X = subset of X_1

or subset of X_1 + extra elt

$= 2^k + 2^k$ subsets

$= 2^{k+1}$.

2 (a) X and Y have the same cardinal number means $\exists$ a 1-1 and onto mapping $f: X \to Y$. X is countable means it has the same cardinal number as $\{1, 2, \ldots, n\}$ some n, or as $\mathbb{N}$.

Write $\mathbb{Q}$ as below & list as below

$$\begin{array}{cccccccc}
1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & \cdots \\
2 & 2/2 & 2/3 & 2/4 & 2/5 & 2/6 & \cdots \\
3 & 3/2 & 3/3 & 3/4 & \cdots \\
\vdots & \vdots & & \text{etc.}
\end{array}$$

Write $(0,1)$ in decimal form.
If they can be listed we have

$.a_{11} a_{12} a_{13} a_{14} \cdots$

$.a_{21} a_{22} a_{23} \cdots$

$\vdots$

$a_{nn} \cdots$

$.a_{n1} a_{n2} a_{n3} \cdots$

Let $b_i = a_{ii} + 1$ if $a_{ii} \neq 9$

$= 0$ if $a_{ii} = 9$.

Then $.b_1 b_2 b_3 \cdots$ is not on the list.

So $(0,1)$ is not countable. Hence $\mathbb{R}$ is not countable

(b) $\leq$ is a partial order on X if it is a relation such that

$x \leq x \quad \forall x \in X$.

$x \leq y$ and $y \leq x \Rightarrow x = y$

$x \leq y$ and $y \leq z \Rightarrow x \leq z$.

A total order is a partial order such that $x \leq y$ or $y \leq x$ $\forall x, y \in X$.

The axiom of choice states that for any collection of set X_α, $\prod_\alpha X_\alpha$ is non empty, i.e we can find a function that picks an elements x_α from each X_α.

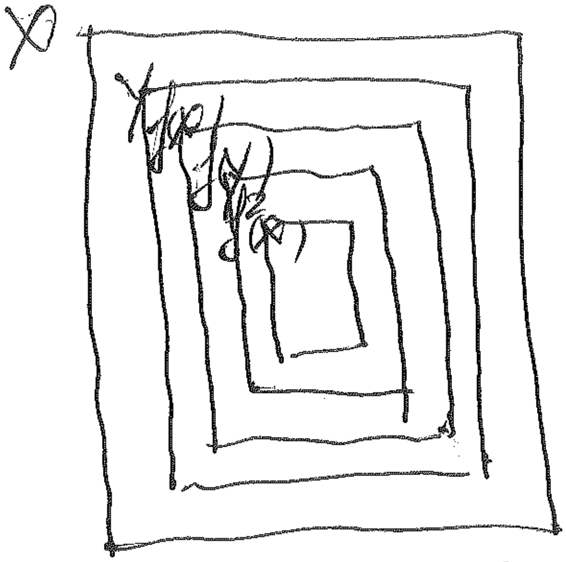
A set is well ordered if in the given ordering every subset has a least element

Given $\{X_\alpha\}$ Let $X = \cup X_\alpha$. Then well order X. Now each X_α has a least element x_α and we can choose that.

(c) The Schröder-Bernstein theorem says that if $\exists f: X \to Y$ 1-1 & $\exists g: Y \to X$ 1-1 then $\exists h: X \to Y$ 1-1 and onto

i.e. $\# X \le \# Y$ and $\# Y \le \# X \Rightarrow \# X = \# Y$.

<u>Proof</u>: first identify Y with the subset $g(Y)$ of X. Then we have the picture



write $X = (X \setminus Y) \cup (Y \setminus f(X)) \cup (f(X) \setminus f(Y)) \cup \cdots \cup \square$

$Y = (Y \setminus f(X)) \cup (f(X) \setminus f(Y)) \cup (f(Y) \setminus f(f(X))) \cup \cdots \cup \square$

define $h = f$ on $(X \setminus Y) \cup (f(X) \setminus f(Y)) \cdots$

$= \mathrm{id}$ on $(Y \setminus f(X)) \cup f(Y) \setminus f(f(X)) \cdots \cup \square$

Note f is 1-1 $\Rightarrow$ $f(X \setminus Y) = f(X) \setminus f(Y)$

and all the pieces are disjoint

$\Rightarrow$ h is 1-1.

Cantor's Theorem says $\# P(X) > \# X$.

i.e. $\nexists\, f$ 1-1, onto $X \to P(X)$

suppose such f exists.

Let $A = \{x : x \notin f(x)\}$.

then $\exists\, x_0$ s.t. $f(x_0) = A$.

If $x_0 \in A$, then $x \notin f(x_0) = A$ $\Rightarrow \Leftarrow$.

If $x_0 \notin A$, then $x_0 \notin f(x_0)$ $\therefore x_0 \in A$ $\Rightarrow \Leftarrow$.

3 (a) O_n open $\not\Rightarrow \cap O_n$ open n infinite

Let $O_n = (-\frac{1}{n}, \frac{1}{n})$ $\cap O_n = \{0\}$ not open

O_n open $\rightarrow \cup O_n$ open

Let $x \in \cup O_n \rightarrow x \in O_{n_i}$ some n_i

$\Rightarrow \exists \epsilon$ s.t. $N(x, \epsilon) \subset O_{n_i}$

$\Rightarrow N(x, \epsilon) \subset \cup O_n$

$\Rightarrow \cup O_n$ open.

O_n closed $\rightarrow \cap O_n$ closed.

O_n closed $\rightarrow O_n^c$ open

$\rightarrow \cup O_n^c$ open by above

$\rightarrow (\cup O_n^c)^c$ closed.

$\rightarrow \cap O_n$ closed

O_n closed $\not\Rightarrow \cup O_n$ closed.

$O_n = \{\frac{1}{n}\}$ $\cup O_n$ not closed

since 0 is an accumulation point not in $\cup O_n$.

(b) $A \subset \mathbb{R}$ A not bounded.

Let $O_n = (-n, n)$. then $A \subset \cup O_n = \mathbb{R}$

But a finite number cannot cover A.

3(b) Let $A \subset \mathbb{R}$ not closed

$\exists\ x$ an acc. point of A not in A

Let $O_n = (-\infty, x - \frac{1}{n}) \cup (x + 1/n, +\infty)$

then $A \subset \bigcup O_n = \mathbb{R} \setminus \{x\}$

But a finite number cannot have points arbitrarily close to x.

Now let $A \subset \mathbb{R}$ closed + bounded

Case 1 $A = [a, b]$.

Suppose $A \subset \bigcup O_\alpha$ but a finite number of O_α don't cover $A = I_1$

Consider $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$. If a finite number of O_α cover each of these, then a finite number cover A. So one, call it I_2 does not have a finite subcover. Divide I_2 into two halves and proceed as before. We get

$I_1 \supset I_2 \supset \quad \supset I_n \supset \cdots$

where each I_n is a closed

bounded interval which is not contained in any finite union of the O_α's.

By the Nested Intervals Theorem $\cap I_n \neq \phi$. Let $x \in \cap I_n$. Then $x \in A$ so $x \in O_{\alpha_i}$ some α_i and O_{α_i} is open so $\exists \in > 0$ s.t $N(x, \in) \subset O_{\alpha_i}$. But $|I_n| \to 0$, and if $|I_n| < \in$, then $x \in I_n \to N(x, \in) \subset O_{\alpha_i}$ and therefore $I_n \subset O_{\alpha_i}$. But then the I_n have a finite subcover. So our original hypothesis that $[a, b]$ does not have a finite subcover has led to a contradiction. Hence our hypothesis was false.

<u>Case 2</u> If A is simply closed + bounded if $A \subset \cup O_\alpha$, $\exists [a, b]$ s.t $A \subset [a, b]$ and if we consider all O_α's and A^c then we have an open cover for ~~A~~ $[a, b]$. By Case 1

3. $\exists$ $O_{\alpha_1}, \dots, O_{\alpha_n}$ and possibly A^c which cover $[a, b]$.

Then $O_{\alpha_1}, \dots, O_{\alpha}$ and possibly A^c cover A, But $A^c \cap A = \emptyset$, so $O_{\alpha_1}, \dots, O_{\alpha_n}$ cover A.

4 (a) f is cont. on $\mathbb{R}$ if

$\forall x_0 \in \mathbb{R}$, $\forall \epsilon$, $\exists \delta$ s.t.

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

i.e. $\forall x_0 \in \mathbb{R}$ $\forall N(f(x_0), \epsilon)$ $\exists N(x_0, \delta)$

~~s.t. x ∈ N~~ $f(N(x_0, \delta)) \subset N(f(x_0), \epsilon)$.

Let O open in $\mathbb{R}$.

Let $x_0 \in f^{-1}(O)$. then $f(x_0) \in O$

$\therefore$ $\exists$ ϵ. s.t. $N(f(x_0), \epsilon) \subset O$.

By above $\Rightarrow$ $\exists$ $N(x_0, \delta)$ s.t. $f(N(x_0, \delta)) \subset N(f(x_0), \epsilon)$

$\therefore$ $N(x_0, \delta) \subset f^{-1}(O)$

$\therefore$ x_0 is an interior point of $f^{-1}(O)$, so $f^{-1}(O)$ is open.

4 (a) Conversely, if $f^{-1}(O)$ open $\forall$ O open.

Given $N(f(x_0), \epsilon))$ this is open

so $f^{-1}(N(f(x_0), \epsilon))$ is open.

$x_0 \in f^{-1}(N(f(x_0), \epsilon))$, so

$\exists\, \delta$ s.t. $N(x_0, \delta) \subset f^{-1}(N(f(x_0), \epsilon))$

$\Rightarrow f(N(x_0, \delta)) \subset N(f(x_0), \epsilon)$

as desired.

(b) $A \subset \mathbb{R}$ compact, f cont.

Suppose $f(A) \subset \bigcup O_\alpha$ O_α open.

then $A \subset f^{-1}(\bigcup O_\alpha) = \bigcup f^{-1}(O_\alpha)$

and $f^{-1}(O_\alpha)$ is open. so A

compact $\Rightarrow A \subset f^{-1}(O_{\alpha_1}) \cup \cdots \cup f^{-1}(O_{\alpha_n})$

$\Rightarrow f(A) \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$

Hence $f(A)$ is compact.

(c) Suppose f is cont. on $[a,b]$.

then $f([a,b])$ is compact by (b)

$\Rightarrow$ closed + bounded by (a)

bdd $\Rightarrow$ l.u.b & g.l.b. exist.

closed $\Rightarrow$ belong to $f([a,b])$.

So $\exists$ x_0, and x_1 in $[a,b]$ such that

$$f(x_0) = \max_{x\in[a,b]} f(x) \quad + \quad f(x_1) = \min_{x\in[a,b]} f(x).$$

This is the Extreme Value Theorem.