Assignment 9.

Q1 Solve $(D+2)(D+3)y = 2e^{2x}$.

Q2 Solve $(D+1)Dy = e^{x}\cos 2x$.

Q3 Solve $(D+2)^3 y = 4e^{-2x}$

Do the marked questions on the next pages

p 576/7 32, 34, 46

p 594/5 20, 26, 28.

(c) How many cells are present after 2 hours?
(d) How long does it take for the number of cells to reach 1,000,000?

31. Radon-222 is a radioactive gas with a half-life of 3.83 days. This gas is a health hazard because it tends to get trapped in the basements of houses, and many health officials suggest that homeowners seal their basements to prevent entry of the gas. Assume that 5.0×10^7 radon atoms are trapped in a basement at the time it is sealed and that $y(t)$ is the number of atoms present t days later.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.
(c) How many atoms will be present after 30 days?
(d) How long will it take for 90% of the original quantity of gas to decay?

32. Polonium-210 is a radioactive element with a half-life of 140 days. Assume that 20 milligrams of the element are placed in a lead container and that $y(t)$ is the number of milligrams present t days later.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.
(c) How many milligrams will be present after 10 weeks?
(d) How long will it take for 70% of the original sample to decay?

33. Suppose that 200 fruit flies are placed in a breeding container that can support at most 10,000 flies. Assuming that the population grows exponentially at a rate of 2% per day, how long will it take for the container to reach capacity?

34. Suppose that the town of Grayrock had a population of 10,000 in 2006 and a population of 12,000 in 2011. Assuming an exponential growth model, in what year will the population reach 20,000?

35. A scientist wants to determine the half-life of a certain radioactive substance. She determines that in exactly 5 days a 10.0-milligram sample of the substance decays to 3.5 milligrams. Based on these data, what is the half-life?

36. Suppose that 40% of a certain radioactive substance decays in 5 years.
(a) What is the half-life of the substance in years?
(b) Suppose that a certain quantity of this substance is stored in a cave. What percentage of it will remain after t years?

FOCUS ON CONCEPTS

37. (a) Make a conjecture about the effect on the graphs of $y = y_0e^{kt}$ and $y = y_0e^{-kt}$ of varying k and keeping y_0 fixed. Confirm your conjecture with a graphing utility.
(b) Make a conjecture about the effect on the graphs of $y = y_0e^{kt}$ and $y = y_0e^{-kt}$ of varying y_0 and keeping k fixed. Confirm your conjecture with a graphing utility.

38. (a) What effect does increasing y_0 and keeping k fixed have on the doubling time or half-life of an exponential model? Justify your answer.
(b) What effect does increasing k and keeping y_0 fixed have on the doubling time and half-life of an exponential model? Justify your answer.

39. (a) There is a trick, called the ***Rule of 70***, that can be used to get a quick estimate of the doubling time or half-life of an exponential model. According to this rule, the doubling time or half-life is roughly 70 divided by the percentage growth or decay rate. For example, we showed in Example 5 that with a continued growth rate of 1.10% per year the world population would double every 63 years. This result agrees with the Rule of 70, since $70/1.10 \approx 63.6$. Explain why this rule works.
(b) Use the Rule of 70 to estimate the doubling time of a population that grows exponentially at a rate of 2% per year.
(c) Use the Rule of 70 to estimate the half-life of a population that decreases exponentially at a rate of 3.5% per hour.
(d) Use the Rule of 70 to estimate the growth rate that would be required for a population growing exponentially to double every 10 years.

40. Find a formula for the tripling time of an exponential growth model.

41. In 1950, a research team digging near Folsom, New Mexico, found charred bison bones along with some leaf-shaped projectile points (called the "Folsom points") that had been made by a Paleo-Indian hunting culture. It was clear from the evidence that the bison had been cooked and eaten by the makers of the points, so that carbon-14 dating of the bones made it possible for the researchers to determine when the hunters roamed North America. Tests showed that the bones contained between 27% and 30% of their original carbon-14. Use this information to show that the hunters lived roughly between 9000 B.C. and 8000 B.C.

42. (a) Use a graphing utility to make a graph of p_{rem} versus t, where p_{rem} is the percentage of carbon-14 that remains in an artifact after t years.
(b) Use the graph to estimate the percentage of carbon-14 that would have to have been present in the 1988 test of the Shroud of Turin for it to have been the burial shroud of Jesus of Nazareth (see Example 7).

43. (a) It is currently accepted that the half-life of carbon-14 might vary ± 40 years from its nominal value of 5730 years. Does this variation make it possible that the Shroud of Turin dates to the time of Jesus of Nazareth (see Example 7)?
(b) Review the subsection of Section 2.9 entitled Error Propagation, and then estimate the percentage error that

results in the computed age of an artifact from an $r\%$ error in the half-life of carbon-14.

44. Suppose that a quantity y has an exponential growth model $y = y_0e^{kt}$ or an exponential decay model $y = y_0e^{-kt}$, and it is known that $y = y_1$ if $t = t_1$. In each case find a formula for k in terms of y_0, y_1, and t_1, assuming that $t_1 \neq 0$.

45. (a) Show that if a quantity $y = y(t)$ has an exponential model, and if $y(t_1) = y_1$ and $y(t_2) = y_2$, then the doubling time or the half-life T is

$$T = \left|\frac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)}\right|$$

(b) In a certain 1-hour period the number of bacteria in a colony increases by 30%. Assuming an exponential growth model, what is the doubling time for the colony?

46. Suppose that P dollars is invested at an annual interest rate of $r \times 100\%$. If the accumulated interest is credited to the account at the end of the year, then the interest is said to be *compounded annually*; if it is credited at the end of each 6-month period, then it is said to be *compounded semiannually*; and if it is credited at the end of each 3-month period, then it is said to be *compounded quarterly*. The more frequently the interest is compounded, the better it is for the investor since more of the interest is itself earning interest.

(a) Show that if interest is compounded n times a year at equally spaced intervals, then the value A of the investment after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

(b) One can imagine interest to be compounded each day, each hour, each minute, and so forth. Carried to the limit one can conceive of interest compounded at each instant of time; this is called ***continuous compounding***. Thus, from part (a), the value A of P dollars after t years when invested at an annual rate of $r \times 100\%$, compounded continuously, is

$$A = \lim_{n \to +\infty} P\left(1 + \frac{r}{n}\right)^{nt}$$

Use the fact that $\lim_{x\to 0}(1 + x)^{1/x} = e$ to prove that $A = Pe^{rt}$.

(c) Use the result in part (b) to show that money invested at continuous compound interest increases at a rate proportional to the amount present.

47. (a) If \$1000 is invested at 7% per year compounded continuously (Exercise 46), what will the investment be worth after 5 years?

(b) If it is desired that an investment at 7% per year compounded continuously should have a value of \$10,000 after 10 years, how much should be invested now?

(c) How long does it take for an investment at 7% per year compounded continuously to double in value?

48. What is the effective annual interest rate for an interest rate of $r\%$ per year compounded continuously?

49. Assume that $y = y(t)$ satisfies the logistic equation with $y_0 = y(0)$ the initial value of y.

(a) Use separation of variables to derive the solution

$$y = \frac{y_0L}{y_0 + (L - y_0)e^{-kt}}$$

(b) Use part (a) to show that $\lim_{t\to+\infty} y(t) = L$.

50. Use your answer to Exercise 49 to derive a solution to the model for the spread of disease [Equation (6) of Section 8.1].

51. The graph of a solution to the logistic equation is known as a ***logistic curve***, and if $y_0 > 0$, it has one of four general shapes, depending on the relationship between y_0 and L. In each part, assume that $k = 1$ and use a graphing utility to plot a logistic curve satisfying the given condition.

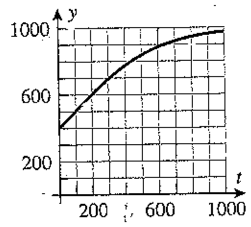
(a) $y_0 > L$ (b) $y_0 = L$

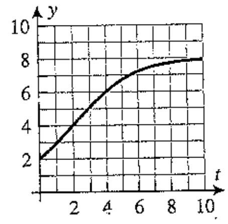
(c) $L/2 \leq y_0 < L$ (d) $0 < y_0 < L/2$

52–53 The graph of a logistic model

$$y = \frac{y_0L}{y_0 + (L - y_0)e^{-kt}}$$

is shown. Estimate y_0, L, and k.

52.


53.


54. Plot a solution to the initial-value problem

$$\frac{dy}{dt} = 0.98\left(1 - \frac{y}{5}\right)y, \quad y_0 = 1$$

55. Suppose that the growth of a population $y = y(t)$ is given by the logistic equation

$$y = \frac{60}{5 + 7e^{-t}}$$

(a) What is the population at time $t = 0$?

(b) What is the carrying capacity L?

(c) What is the constant k?

(d) When does the population reach half of the carrying capacity?

(e) Find an initial-value problem whose solution is $y(t)$.

56. Suppose that the growth of a population $y = y(t)$ is given by the logistic equation

$$y = \frac{500}{1 + 499e^{-0.9t}}$$

(a) What is the population at time $t = 0$?

(b) What is the carrying capacity L?

(c) What is the constant k?

(cont.)

30. (a) Prove that solutions need not be unique for nonlinear initial-value problems by finding two solutions to

$$y\frac{dy}{dx} = x, \quad y(0) = 0$$

(b) Prove that solutions need not exist for nonlinear initial-value problems by showing that there is no solution for

$$y\frac{dy}{dx} = -x, \quad y(0) = 0$$

31. Writing Explain why the quantity μ in the *Method of Inte grating Factors* is called an "integrating factor" and explai its role in this method.

32. Writing Suppose that a given first-order differential equation can be solved both by the method of integrating factors and by separation of variables. Discuss the advantages and disadvantages of each method.

✔ QUICK CHECK ANSWERS 8.4

1. Step 1: $e^{\int p(x)\,dx}$; Step 2: μy, $\mu q(x)$; Step 3: $\frac{1}{\mu}\int \mu q(x)\,dx$ 2. x 3. $\frac{dy}{dt} + \frac{y}{20} = 15$, $y(0) = 30$

CHAPTER 8 REVIEW EXERCISES

C CAS

1. Give an informal explanation of why one might expect the number of arbitrary constants in the general solution of a differential equation to be equal to the order of the equation.

2. Which of the given differential equations are separable?

(a) $\frac{dy}{dx} = f(x)g(y)$ (b) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$

(c) $\frac{dy}{dx} = f(x) + g(y)$ (d) $\frac{dy}{dx} = \sqrt{f(x)g(y)}$

3–5 Solve the differential equation by the method of separation of variables. ■

3. $\frac{dy}{dx} = (1 + y^2)x^2$ **4.** $4\tan y - \frac{dy}{dx}\sec x = 0$

5. $(1 + y^2)y' = e^x y$

6–8 Solve the initial-value problem by the method of separation of variables. ■

6. $y' = 1 + y^2$, $y(0) = \sqrt{3}$ **7.** $y' = \frac{y^5}{x(1 + y^4)}$, $y(1) = 1$

8. $y' = 4y^2\sec^2 2x$, $y(\pi/8) = 1/2$

9. Sketch the integral curve of $y' = -2xy^2$ that passes through the point $(0, 1)$.

10. Sketch the integral curve of $2yy' = 1$ that passes through the point $(0, 1)$ and the integral curve that passes through the point $(0, -1)$.

11. Sketch the slope field for $y' = xy/8$ at the 25 gridpoints (x, y), where $x = 0, 1, \ldots, 4$ and $y = 0, 1, \ldots, 4$.

12. Solve the differential equation $y' = xy/8$, and find a family of integral curves for the slope field in Exercise 11.

13–14 Use Euler's Method with the given step size Δx to approximate the solution of the initial-value problem over the stated interval. Present your answer as a table and as a graph.

13. $dy/dx = \sqrt{y}$, $y(0) = 1$, $0 \le x \le 4$, $\Delta x = 0.5$

14. $dy/dx = \sin y$, $y(0) = 1$, $0 \le x \le 2$, $\Delta x = 0.5$

15. Consider the initial-value problem

$$y' = \cos 2\pi t, \quad y(0) = 1$$

Use Euler's Method with five steps to approximate $y(1)$.

16. Use Euler's Method with a step size of $\Delta t = 0.1$ to approximate the solution of the initial-value problem

$$y' = 1 + 5t - y, \quad y(1) = 5$$

over the interval $[1, 2]$.

17. Cloth found in an Egyptian pyramid contains 77.5% of its original carbon-14. Estimate the age of the cloth.

18. Suppose that an initial population of 3000 bacteria grows exponentially at a rate of 1% per hour and that $y = y(t)$ is the number of bacteria present after t hours.

(a) Find an initial-value problem whose solution is $y(t)$.

(b) Find a formula for $y(t)$.

(c) What is the doubling time for the population?

(d) How long does it take for the population of bacteria to reach 30,000?

19–20 Solve the differential equation by the method of integrating factors. ■

19. $\frac{dy}{dx} + 4y = e^{-2x}$ **20.** $\frac{dy}{dx} + y - \frac{1}{1 + e^x} = 0$

21–23 Solve the initial-value problem by the method of integrating factors. ■

21. $y' - xy = x,\ y(0) = 5$

22. $xy' + 2y = 4x^2,\ y(1) = 3$

23. $y' \cosh x + y \sinh x = \cosh^2 x,\ y(0) = 2$

24. (a) Solve the initial-value problem
$$y' - y = x \sin 3x, \quad y(0) = 2$$
by the method of integrating factors, using a CAS to perform any difficult integrations.
(b) Use the CAS to solve the initial-value problem directly, and confirm that the answer is consistent with that obtained in part (a).
(c) Graph the solution.

25. Classify the following first-order differential equations as separable, linear, both, or neither.
(a) $\dfrac{dy}{dx} - 3y = \sin x$ (b) $\dfrac{dy}{dx} + xy = x$
(c) $y\dfrac{dy}{dx} - x = 1$ (d) $\dfrac{dy}{dx} + xy^2 = \sin(xy)$

26. Determine whether the methods of integrating factors and separation of variables produce the same solutions of the differential equation
$$\frac{dy}{dx} - 4xy = x$$

27. A tank contains 1000 gal of fresh water. At time $t = 0$ min, brine containing 5 oz of salt per gallon of brine is poured into the tank at a rate of 10 gal/min, and the mixed solution is drained from the tank at the same rate. After 15 min that process is stopped and fresh water is poured into the tank at the rate of 5 gal/min, and the mixed solution is drained from the tank at the same rate. Find the amount of salt in the tank at time $t = 30$ min.

28. Suppose that a room containing 1200 ft^3 of air is free of carbon monoxide. At time $t = 0$ cigarette smoke containing 4% carbon monoxide is introduced at the rate of 0.1 ft^3/min, and the well-circulated mixture is vented from the room at the same rate.
(a) Find a formula for the percentage of carbon monoxide in the room at time t.
(b) Extended exposure to air containing 0.012% carbon monoxide is considered dangerous. How long will it take to reach this level?

Source: This is based on a problem from William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations*, 7th ed., John Wiley & Sons, New York, 2001.

CHAPTER 8 MAKING CONNECTIONS

1. Consider the first-order differential equation
$$\frac{dy}{dx} + py = q$$
where p and q are constants. If $y = y(x)$ is a solution to this equation, define $u = u(x) = q - py(x)$.
(a) Without solving the differential equation, show that u grows exponentially as a function of x if $p < 0$, and decays exponentially as a function of x if $0 < p$.
(b) Use the result of part (a) and Equations (13–14) of Section 8.2 to solve the initial-value problem
$$\frac{dy}{dx} + 2y = 4, \quad y(0) = 1$$

2. Consider a differential equation of the form
$$\frac{dy}{dx} = f(ax + by + c)$$
where f is a function of a single variable. If $y = y(x)$ is a solution to this equation, define $u = u(x) = ax + by(x) + c$.
(a) Find a separable differential equation that is satisfied by the function u.
(b) Use your answer to part (a) to solve
$$\frac{dy}{dx} = \frac{1}{x + y}$$

3. A first-order differential equation is ***homogeneous*** if it can be written in the form
$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{for } x \neq 0$$
where f is a function of a single variable. If $y = y(x)$ is a solution to a first-order homogeneous differential equation, define $u = u(x) = y(x)/x$.
(a) Find a separable differential equation that is satisfied by the function u.
(b) Use your answer to part (a) to solve
$$\frac{dy}{dx} = \frac{x - y}{x + y}$$

4. A first-order differential equation is called a ***Bernoulli equation*** if it can be written in the form
$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad \text{for } n \neq 0, 1$$
If $y = y(x)$ is a solution to a Bernoulli equation, define $u = u(x) = [y(x)]^{1-n}$.
(a) Find a first-order linear differential equation that is satisfied by u.
(b) Use your answer to part (a) to solve the initial-value problem
$$x\frac{dy}{dx} - y = -2xy^2, \quad y(1) = \frac{1}{2}$$