Assignment 9

Q1 Solve $\frac{dy}{dx} + y + \frac{1}{1-e^x} = 0$.

Q2 Solve $(D-1)^3 y = 0$ in 3 steps.

Q3. Show that if B_1 and B_2 are any real numbers, then we can find A_1 and A_2 complex numbers with

$$A_1 + A_2 = B_1$$
$$-iA_1 + iA_2 = B_2.$$

+ textbook questions.

that results from discarding the second term. We call E the *truncation error*.

(a) Approximate the integral in Exercise 64 by applying Simpson's rule with $n = 10$ subdivisions to the integral

$$\int_0^3 e^{-x^2}\,dx$$

Round your answer to four decimal places and compare it to $\frac{1}{2}\sqrt{\pi}$ rounded to four decimal places.

(b) Use the result that you obtained in Exercise 52 and the fact that $e^{-x^2} \le \frac{1}{3}xe^{-x^2}$ for $x \ge 3$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2.1 \times 10^{-5}$.

67. (a) It can be shown that

$$\int_0^{+\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$

Approximate this integral by applying Simpson's rule with $n = 20$ subdivisions to the integral

$$\int_0^4 \frac{1}{x^6+1}\,dx$$

Round your answer to three decimal places and compare it to $\pi/3$ rounded to three decimal places.

(b) Use the result that you obtained in Exercise 52 and the fact that $1/(x^6+1) < 1/x^6$ for $x \ge 4$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2 \times 10^{-4}$.

68. For what values of p does $\int_0^{+\infty} e^{px}\,dx$ converge?

69. Show that $\int_0^1 dx/x^p$ converges if $p < 1$ and diverges if $p \ge 1$.

C 70. It is sometimes possible to convert an improper integral into a "proper" integral having the same value by making an appropriate substitution. Evaluate the following integral by making the indicated substitution, and investigate what happens if you evaluate the integral directly using a CAS.

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx;\quad u = \sqrt{1-x}$$

71–72 Transform the given improper integral into a proper integral by making the stated u-substitution; then approximate the proper integral by Simpson's rule with $n = 10$ subdivisions. Round your answer to three decimal places.

71. $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{x}}\,dx;\quad u = \sqrt{x}$

72. $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{1-x}}\,dx;\quad u = \sqrt{1-x}$

73. **Writing** What is "improper" about an integral over an infinite interval? Explain why Definition 4.5.1 for $\int_a^b f(x)\,dx$ fails for $\int_a^{+\infty} f(x)\,dx$. Discuss a strategy for assigning a value to $\int_a^{+\infty} f(x)\,dx$.

74. **Writing** What is "improper" about a definite integral over an interval on which the integrand has an infinite discontinuity? Explain why Definition 4.5.1 for $\int_a^b f(x)\,dx$ fails if the graph of f has a vertical asymptote at $x = a$. Discuss a strategy for assigning a value to $\int_a^b f(x)\,dx$ in this circumstance.

✔ QUICK CHECK ANSWERS 7.8

1. (a) proper (b) improper, since $\cot x$ has an infinite discontinuity at $x = \pi$ (c) improper, since there is an infinite interval of integration (d) improper, since there is an infinite interval of integration and the integrand has an infinite discontinuity at $x = 1$
2. (b) $\displaystyle\lim_{b\to\pi^-}\int_{\pi/4}^{b} \cot x\,dx$ (c) $\displaystyle\lim_{b\to+\infty}\int_0^b \frac{1}{x^2+1}\,dx$ (d) $\displaystyle\lim_{a\to1^+}\int_a^2 \frac{1}{x^2-1}\,dx + \lim_{b\to+\infty}\int_2^b \frac{1}{x^2-1}\,dx$ 3. $\dfrac{1}{p-1}$; $p > 1$
4. (a) 1 (b) diverges (c) diverges (d) 3

CHAPTER 7 REVIEW EXERCISES

1–6 Evaluate the given integral with the aid of an appropriate u-substitution.

1. $\displaystyle\int \sqrt{4+5x}\,dx$
2. $\displaystyle\int \frac{1}{\sec \pi x}\,dx$
3. $\displaystyle\int \sqrt{\cos x}\sin x\,dx$
4. $\displaystyle\int \frac{dx}{x\ln x}$
5. $\displaystyle\int x\tan^2(x^2)\sec^2(x^2)\,dx$
6. $\displaystyle\int_0^4 \frac{\sqrt{x}}{x+4}\,dx$

7. (a) Evaluate the integral

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx$$

three ways: using the substitution $u = \sqrt{x}$, using the substitution $u = \sqrt{2-x}$, and completing the square.

(b) Show that the answers in part (a) are equivalent.

8. Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$
(a) using integration by parts
(b) using the substitution $u = \sqrt{x^2+1}$.

9–12 Use integration by parts to evaluate the integral.

9. $\int xe^{-x}\,dx$
10. $\int x \sin 3x\,dx$
11. $\int \ln(2x+7)\,dx$
12. $\int_0^{1/2} \tan^{-1}(2x)\,dx$

13. Evaluate $\int 8x^4 \cos 2x\,dx$ using tabular integration by parts.

14. A particle moving along the x-axis has velocity function $v(t) = t^2e^{-t}$. How far does the particle travel from time $t = 0$ to $t = 4$?

15–20 Evaluate the integral.

15. $\int \sin^2 6\theta\,d\theta$
16. $\int \sin^3 2x \cos^2 2x\,dx$
17. $\int \sin x \cos 3x\,dx$
18. $\int_0^{\pi/6} \sin 2x \cos 4x\,dx$
19. $\int \sin^4 2x\,dx$
20. $\int x \cos^5(x^2)\,dx$

21–26 Evaluate the integral by making an appropriate trigonometric substitution.

21. $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$
22. $\int \frac{dx}{x^2\sqrt{4-x^2}}$
23. $\int \frac{dx}{\sqrt{x^2-1}}$
24. $\int \frac{x^2}{\sqrt{x^2-16}}\,dx$
25. $\int \frac{x^2}{\sqrt{9+x^2}}\,dx$
26. $\int \frac{\sqrt{1+4x^2}}{x}\,dx$

27–32 Evaluate the integral using the method of partial fractions.

27. $\int \frac{dx}{x^2+4x-5}$
28. $\int \frac{dx}{x^2+7x+6}$
29. $\int \frac{x^2+2}{x+2}\,dx$
30. $\int \frac{x^2+x-16}{(x-1)(x-3)^2}\,dx$
31. $\int \frac{x^2}{(x+2)^3}\,dx$
32. $\int \frac{dx}{x^3+x}$

33. Consider the integral $\int \frac{1}{x^3-x}\,dx$.
(a) Evaluate the integral using the substitution $x = \sec\theta$. For what values of x is your result valid?
(b) Evaluate the integral using the substitution $x = \sin\theta$. For what values of x is your result valid?
(c) Evaluate the integral using the method of partial fractions. For what values of x is your result valid?

34. Find the area of the region that is enclosed by the curves $y = (x-3)/(x^3+x^2)$, $y = 0$, $x = 1$, and $x = 3$.

35–40 Use the Integral Table to evaluate the integral.

35. $\int \sin 7x \cos 10x\,dx$
36. $\int (x^3 - x^2)e^{-x}\,dx$
37. $\int x\sqrt{x-x^2}\,dx$
38. $\int \frac{dx}{x\sqrt{4x+3}}$
39. $\int \tan^2 2x\,dx$
40. $\int \frac{3x-1}{2+x^2}\,dx$

41–42 Approximate the integral using (a) the midpoint approximation M_{10}, (b) the trapezoidal approximation T_{10}, and (c) Simpson's rule approximation S_{20}. In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places.

41. $\int_1^3 \frac{1}{\sqrt{x+1}}\,dx$
42. $\int_{-2}^2 \frac{1}{1+x^2}\,dx$

43–44 Use inequalities (12), (13), and (14) of Section 7.7 to find upper bounds on the errors in parts (a), (b), or (c) of the indicated exercise.

43. Exercise 41
44. Exercise 42

45–46 Use inequalities (12), (13), and (14) of Section 7.7 to find a number n of subintervals for (a) the midpoint approximation M_n, (b) the trapezoidal approximation T_n, and (c) Simpson's rule approximation S_n to ensure the absolute error will be less than 10^{-4}.

45. Exercise 41
46. Exercise 42

47–50 Evaluate the integral if it converges.

47. $\int_0^{+\infty} e^{-x}\,dx$
48. $\int_{-\infty}^2 \frac{dx}{x^2+4}$
49. $\int_0^9 \frac{dx}{\sqrt{9-x}}$
50. $\int_0^1 \frac{1}{2x-1}\,dx$

51. Find the area that is enclosed between the x-axis and the curve $y = (\ln x - 1)/x^2$ for $x \geq e$.

52. Find the volume of the solid that is generated when the region between the x-axis and the curve $y = e^{-x}$ for $x \geq 0$ is revolved about the y-axis.

53. Find a positive value of a that satisfies the equation

$$\int_0^{+\infty} \frac{1}{x^2+a^2}\,dx = 1$$

54. Consider the following methods for evaluating integrals: u-substitution, integration by parts, partial fractions, reduction formulas, and trigonometric substitutions. In each part, state the approach that you would try first to evaluate the integral. If none of them seems appropriate, then say so. You need not evaluate the integral.

(a) $\int x \sin x\,dx$
(b) $\int \cos x \sin x\,dx$

(cont.)

(c) $\int \tan^7 x\, dx$
(d) $\int \tan^7 x \sec^2 x\, dx$
(e) $\int \frac{3x^2}{x^3+1}\, dx$
(f) $\int \frac{3x^2}{(x+1)^3}\, dx$
(g) $\int \tan^{-1} x\, dx$
(h) $\int \sqrt{4-x^2}\, dx$
(i) $\int x\sqrt{4-x^2}\, dx$

55–74 Evaluate the integral. ■

55. $\int \frac{dx}{(3+x^2)^{3/2}}$

56. $\int x \cos 4x\, dx$

57. $\int_0^{\pi/4} \tan^7 \theta\, d\theta$

58. $\int \frac{\cos\theta}{\sin^2\theta - 6\sin\theta + 12}\, d\theta$

59. $\int \sin^2 2x \cos^3 2x\, dx$

60. $\int_0^4 \frac{1}{(x-2)^2}\, dx$

61. $\int e^{2x} \cos 3x\, dx$

62. $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx$

63. $\int \frac{dx}{(x-1)(x+2)(x-3)}$

64. $\int_0^{1/3} \frac{dx}{(4-9x^2)^2}$

65. $\int_4^8 \frac{\sqrt{x-4}}{x}\, dx$

66. $\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

67. $\int \frac{1}{\sqrt{e^x+1}}\, dx$

68. $\int \frac{dx}{x(x^2+x+1)}$

69. $\int_0^{1/2} \sin^{-1} x\, dx$

70. $\int \tan^5 4x \sec^4 4x\, dx$

71. $\int \frac{x+3}{\sqrt{x^2+2x+2}}\, dx$

72. $\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta}\, d\theta$

73. $\int_a^{+\infty} \frac{x}{(x^2+1)^2}\, dx$

74. $\int_0^{+\infty} \frac{dx}{a^2+b^2x^2}, \quad a, b > 0$

CHAPTER 7 MAKING CONNECTIONS   C CAS

1. Recall from Theorem 6.3.1 and the discussion preceding it that if $f'(x) > 0$, then the function f is increasing and has an inverse function. Parts (a), (b), and (c) of this problem show that if this condition is satisfied and if f' is continuous, then a definite integral of f^{-1} can be expressed in terms of a definite integral of f.
   (a) Use integration by parts to show that
   $$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b xf'(x)\,dx$$
   (b) Use the result in part (a) to show that if $y = f(x)$, then
   $$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$
   (c) Show that if we let $\alpha = f(a)$ and $\beta = f(b)$, then the result in part (b) can be written as
   $$\int_\alpha^\beta f^{-1}(x)\,dx = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\,dx$$

2. In each part, use the result in Exercise 1 to obtain the equation, and then confirm that the equation is correct by performing the integrations.
   (a) $\int_0^{1/2} \sin^{-1} x\,dx = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \int_0^{\pi/6} \sin x\,dx$
   (b) $\int_e^{e^2} \ln x\,dx = (2e^2 - e) - \int_1^2 e^x\,dx$

3. The ***Gamma function***, $\Gamma(x)$, is defined as
   $$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt$$
   It can be shown that this improper integral converges if and only if $x > 0$.
   (a) Find $\Gamma(1)$.
   (b) Prove: $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. [*Hint:* Use integration by parts.]
   (c) Use the results in parts (a) and (b) to find $\Gamma(2)$, $\Gamma(3)$, and $\Gamma(4)$; and then make a conjecture about $\Gamma(n)$ for positive integer values of n.
   (d) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [*Hint:* See Exercise 64 of Section 7.8.]
   (e) Use the results obtained in parts (b) and (d) to show that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$.

4. Refer to the Gamma function defined in Exercise 3 to show that
   (a) $\int_0^1 (\ln x)^n\,dx = (-1)^n\Gamma(n+1), \quad n > 0$
   [*Hint:* Let $t = -\ln x$.]
   (b) $\int_0^{+\infty} e^{-x^n}\,dx = \Gamma\left(\frac{n+1}{n}\right), \quad n > 0.$
   [*Hint:* Let $t = x^n$. Use the result in Exercise 3(b).]

C 5. A ***simple pendulum*** consists of a mass that swings in a vertical plane at the end of a massless rod of length L, as shown in the accompanying figure. Suppose that a simple pendulum is displaced through an angle θ_0 and released from rest. It can be

The initial-value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = 1$$

has solution $y(x) =$ ________.

ERCISE SET 8.2 Graphing Utility C CAS

10 Solve the differential equation by separation of variables. here reasonable, express the family of solutions as explicit nctions of x.

1. $\frac{dy}{dx} = \frac{y}{x}$
2. $\frac{dy}{dx} = 4(1+y^2)x$
3. $\frac{\sqrt{1+x^2}}{1+y}\frac{dy}{dx} = -x$
4. $(1+x^4)\frac{dy}{dx} = \frac{x^3}{y}$
5. $(2+2y^2)y' = e^x y$
6. $y' = -x^2 y$
7. $e^{-y}\sin x - y'\cos^2 x = 0$
8. $y' - (1+x)(1+y^2) = 0$
9. $\frac{dy}{dx} - \frac{y^2 - y}{\sin x} = 0$
10. $y - \frac{dy}{dx}\csc x = 0$

1–14 Solve the initial-value problem by separation of variables.

11. $y' = \frac{3x^2}{2y + \cos y}, \quad y(0) = \pi$
12. $y' - xe^y = 2e^y, \quad y(0) = 0$
13. $\frac{dy}{dt} = \frac{2t+1}{2y-2}, \quad y(0) = 1$
14. $y'\cosh^2 x - y\cosh 2x = 0, \quad y(0) = 5$

15. (a) Sketch some typical integral curves of the differential equation $y' = y/2x$.
(b) Find an equation for the integral curve that passes through the point $(3, 1)$.

16. (a) Sketch some typical integral curves of the differential equation $y' = -x/y$.
(b) Find an equation for the integral curve that passes through the point $(3, 4)$.

17–18 Solve the differential equation and then use a graphing utility to generate five integral curves for the equation.

17. $(x^2+4)\frac{dy}{dx} + xy = 0$
18. $(\cos y)y' = \cos x$

19–20 Solve the differential equation. If you have a CAS with implicit plotting capability, use the CAS to generate five integral curves for the equation.

19. $y' = \frac{x^2}{1-y^2}$
20. $y' = \frac{y}{1+y^2}$

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. Every differential equation of the form $y' = f(y)$ is separable.

22. If a population is growing exponentially, then the time it takes the population to quadruple is independent of the size of the population.

23. If a radioactive element has a half-life of 1 minute, and if a container holds 32 g of the element at 1:00 P.M., then the amount remaining at 1:05 P.M. will be 1 g.

24. A differential equation of the form

$$h(x)\frac{dy}{dx} = g(y)$$

is not separable.

25. Suppose that the initial condition in Example 1 had been $y(0) = 0$. Show that none of the solutions generated in Example 1 satisfy this initial condition, and then solve the initial-value problem

$$\frac{dy}{dx} = -4xy^2, \quad y(0) = 0$$

Why does the method of Example 1 fail to produce this particular solution?

26. Find all ordered pairs (x_0, y_0) such that if the initial condition in Example 1 is replaced by $y(x_0) = y_0$, the solution of the resulting initial-value problem is defined for all real numbers.

27. Find an equation of a curve with x-intercept 4 whose tangent line at any point (x, y) has slope xe^{-y}.

28. Use a graphing utility to generate a curve that passes through the point $(1, 1)$ and whose tangent line at (x, y) is perpendicular to the line through (x, y) with slope $-2y/(3x^2)$.

29. Suppose that an initial population of 20,000 bacteria grows exponentially at a rate of 2% per hour and that $y = y(t)$ is the number of bacteria present t hours later.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.
(c) How long does it take for the initial population of bacteria to double?
(d) How long does it take for the population of bacteria to reach 45,000?

30. A cell of the bacterium *E. coli* divides into two cells every 20 minutes when placed in a nutrient culture. Let $y = y(t)$ be the number of cells that are present t minutes after a single cell is placed in the culture. Assume that the growth of the bacteria is approximated by an exponential growth model.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.

(cont.)

(c) How many cells are present after 2 hours?
(d) How long does it take for the number of cells to reach 1,000,000?

31. Radon-222 is a radioactive gas with a half-life of 3.83 days. This gas is a health hazard because it tends to get trapped in the basements of houses, and many health officials suggest that homeowners seal their basements to prevent entry of the gas. Assume that 5.0×10^7 radon atoms are trapped in a basement at the time it is sealed and that $y(t)$ is the number of atoms present t days later.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.
(c) How many atoms will be present after 30 days?
(d) How long will it take for 90% of the original quantity of gas to decay?

32. Polonium-210 is a radioactive element with a half-life of 140 days. Assume that 20 milligrams of the element are placed in a lead container and that $y(t)$ is the number of milligrams present t days later.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.
(c) How many milligrams will be present after 10 weeks?
(d) How long will it take for 70% of the original sample to decay?

33. Suppose that 200 fruit flies are placed in a breeding container that can support at most 10,000 flies. Assuming that the population grows exponentially at a rate of 2% per day, how long will it take for the container to reach capacity?

34. Suppose that the town of Grayrock had a population of 10,000 in 2006 and a population of 12,000 in 2011. Assuming an exponential growth model, in what year will the population reach 20,000?

35. A scientist wants to determine the half-life of a certain radioactive substance. She determines that in exactly 5 days a 10.0-milligram sample of the substance decays to 3.5 milligrams. Based on these data, what is the half-life?

36. Suppose that 40% of a certain radioactive substance decays in 5 years.
(a) What is the half-life of the substance in years?
(b) Suppose that a certain quantity of this substance is stored in a cave. What percentage of it will remain after t years?

FOCUS ON CONCEPTS

37. (a) Make a conjecture about the effect on the graphs of $y = y_0e^{kt}$ and $y = y_0e^{-kt}$ of varying k and keeping y_0 fixed. Confirm your conjecture with a graphing utility.
(b) Make a conjecture about the effect on the graphs of $y = y_0e^{kt}$ and $y = y_0e^{-kt}$ of varying y_0 and keeping k fixed. Confirm your conjecture with a graphing utility.

38. (a) What effect does increasing y_0 and keeping k fixed have on the doubling time or half-life of an exponential model? Justify your answer.
(b) What effect does increasing k and keeping y_0 fixed have on the doubling time and half-life of an exponential model? Justify your answer.

39. (a) There is a trick, called the ***Rule of 70***, that can be used to get a quick estimate of the doubling time or half-life of an exponential model. According to this rule, the doubling time or half-life is roughly 70 divided by the percentage growth or decay rate. For example, we showed in Example 5 that with a continued growth rate of 1.10% per year the world population would double every 63 years. This result agrees with the Rule of 70, since $70/1.10 \approx 63.6$. Explain why this rule works.
(b) Use the Rule of 70 to estimate the doubling time of a population that grows exponentially at a rate of 2% per year.
(c) Use the Rule of 70 to estimate the half-life of a population that decreases exponentially at a rate of 3.5% per hour.
(d) Use the Rule of 70 to estimate the growth rate that would be required for a population growing exponentially to double every 10 years.

40. Find a formula for the tripling time of an exponential growth model.

41. In 1950, a research team digging near Folsom, New Mexico, found charred bison bones along with some leaf-shaped projectile points (called the "Folsom points") that had been made by a Paleo-Indian hunting culture. It was clear from the evidence that the bison had been cooked and eaten by the makers of the points, so that carbon-14 dating of the bones made it possible for the researchers to determine when the hunters roamed North America. Tests showed that the bones contained between 27% and 30% of their original carbon-14. Use this information to show that the hunters lived roughly between 9000 B.C. and 8000 B.C.

42. (a) Use a graphing utility to make a graph of p_{rem} versus t, where p_{rem} is the percentage of carbon-14 that remains in an artifact after t years.
(b) Use the graph to estimate the percentage of carbon-14 that would have to have been present in the 1988 test of the Shroud of Turin for it to have been the burial shroud of Jesus of Nazareth (see Example 7).

43. (a) It is currently accepted that the half-life of carbon-14 might vary ± 40 years from its nominal value of 5730 years. Does this variation make it possible that the Shroud of Turin dates to the time of Jesus of Nazareth (see Example 7)?
(b) Review the subsection of Section 2.9 entitled Error Propagation, and then estimate the percentage error that