that results from discarding the second term. We call E the *truncation error*.

(a) Approximate the integral in Exercise 64 by applying Simpson's rule with $n = 10$ subdivisions to the integral

$$\int_0^3 e^{-x^2}\,dx$$

Round your answer to four decimal places and compare it to $\frac{1}{2}\sqrt{\pi}$ rounded to four decimal places.

(b) Use the result that you obtained in Exercise 52 and the fact that $e^{-x^2} \leq \frac{1}{3}xe^{-x^2}$ for $x \geq 3$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2.1 \times 10^{-5}$.

67. (a) It can be shown that

$$\int_0^{+\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$

Approximate this integral by applying Simpson's rule with $n = 20$ subdivisions to the integral

$$\int_0^4 \frac{1}{x^6+1}\,dx$$

Round your answer to three decimal places and compare it to $\pi/3$ rounded to three decimal places.

(b) Use the result that you obtained in Exercise 52 and the fact that $1/(x^6+1) < 1/x^6$ for $x \geq 4$ to show that the truncation error for the approximation in part (a) satisfies $0 < E < 2 \times 10^{-4}$.

68. For what values of p does $\displaystyle\int_0^{+\infty} e^{px}\,dx$ converge?

69. Show that $\displaystyle\int_0^1 dx/x^p$ converges if $p < 1$ and diverges if $p \geq 1$.

C 70. It is sometimes possible to convert an improper integral into a "proper" integral having the same value by making an appropriate substitution. Evaluate the following integral by making the indicated substitution, and investigate what happens if you evaluate the integral directly using a CAS.

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx;\ u = \sqrt{1-x}$$

71–72 Transform the given improper integral into a proper integral by making the stated u-substitution; then approximate the proper integral by Simpson's rule with $n = 10$ subdivisions. Round your answer to three decimal places.

71. $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{x}}\,dx;\ u = \sqrt{x}$

72. $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{1-x}}\,dx;\ u = \sqrt{1-x}$

73. **Writing** What is "improper" about an integral over an infinite interval? Explain why Definition 4.5.1 for $\int_a^b f(x)\,dx$ fails for $\int_a^{+\infty} f(x)\,dx$. Discuss a strategy for assigning a value to $\int_a^{+\infty} f(x)\,dx$.

74. **Writing** What is "improper" about a definite integral over an interval on which the integrand has an infinite discontinuity? Explain why Definition 4.5.1 for $\int_a^b f(x)\,dx$ fails if the graph of f has a vertical asymptote at $x = a$. Discuss a strategy for assigning a value to $\int_a^b f(x)\,dx$ in this circumstance.

✔ QUICK CHECK ANSWERS 7.8

1. (a) proper (b) improper, since $\cot x$ has an infinite discontinuity at $x = \pi$ (c) improper, since there is an infinite interval of integration (d) improper, since there is an infinite interval of integration and the integrand has an infinite discontinuity at $x = 1$ 2. (b) $\displaystyle\lim_{b\to\pi^-}\int_{\pi/4}^b \cot x\,dx$ (c) $\displaystyle\lim_{b\to+\infty}\int_0^b \frac{1}{x^2+1}\,dx$ (d) $\displaystyle\lim_{a\to 1^+}\int_a^2 \frac{1}{x^2-1}\,dx + \lim_{b\to+\infty}\int_2^b \frac{1}{x^2-1}\,dx$ 3. $\dfrac{1}{p-1};\ p > 1$
4. (a) 1 (b) diverges (c) diverges (d) 3

CHAPTER 7 REVIEW EXERCISES

1–6 Evaluate the given integral with the aid of an appropriate u-substitution.

1. $\displaystyle\int \sqrt{4+5x}\,dx$

2. $\displaystyle\int \frac{1}{\sec \pi x}\,dx$

3. $\displaystyle\int \sqrt{\cos x}\,\sin x\,dx$

4. $\displaystyle\int \frac{dx}{x\ln x}$

5. $\displaystyle\int x\tan^2(x^2)\sec^2(x^2)\,dx$

6. $\displaystyle\int_0^4 \frac{\sqrt{x}}{x+4}\,dx$

7. (a) Evaluate the integral

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx$$

three ways: using the substitution $u = \sqrt{x}$, using the substitution $u = \sqrt{2-x}$, and completing the square.

(b) Show that the answers in part (a) are equivalent.

8. Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$
   (a) using integration by parts
   (b) using the substitution $u = \sqrt{x^2+1}$.

9–12 Use integration by parts to evaluate the integral.

9. $\int xe^{-x}\,dx$
10. $\int x \sin 3x\,dx$
11. $\int \ln(2x+7)\,dx$
12. $\int_0^{1/2} \tan^{-1}(2x)\,dx$

13. Evaluate $\int 8x^4 \cos 2x\,dx$ using tabular integration by parts.

14. A particle moving along the x-axis has velocity function $v(t) = t^2e^{-t}$. How far does the particle travel from time $t = 0$ to $t = 4$?

15–20 Evaluate the integral.

15. $\int \sin^2 6\theta\,d\theta$
16. $\int \sin^3 2x \cos^2 2x\,dx$
17. $\int \sin x \cos 3x\,dx$
18. $\int_0^{\pi/6} \sin 2x \cos 4x\,dx$
19. $\int \sin^4 2x\,dx$
20. $\int x \cos^5(x^2)\,dx$

21–26 Evaluate the integral by making an appropriate trigonometric substitution.

21. $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$
22. $\int \frac{dx}{x^2\sqrt{4-x^2}}$
23. $\int \frac{dx}{\sqrt{x^2-1}}$
24. $\int \frac{x^2}{\sqrt{x^2-16}}\,dx$
25. $\int \frac{x^2}{\sqrt{9+x^2}}\,dx$
26. $\int \frac{\sqrt{1+4x^2}}{x}\,dx$

27–32 Evaluate the integral using the method of partial fractions.

27. $\int \frac{dx}{x^2+4x-5}$
28. $\int \frac{dx}{x^2+7x+6}$
29. $\int \frac{x^2+2}{x+2}\,dx$
30. $\int \frac{x^2+x-16}{(x-1)(x-3)^2}\,dx$
31. $\int \frac{x^2}{(x+2)^3}\,dx$
32. $\int \frac{dx}{x^3+x}$

33. Consider the integral $\int \frac{1}{x^3-x}\,dx$.
   (a) Evaluate the integral using the substitution $x = \sec\theta$. For what values of x is your result valid?
   (b) Evaluate the integral using the substitution $x = \sin\theta$. For what values of x is your result valid?
   (c) Evaluate the integral using the method of partial fractions. For what values of x is your result valid?

34. Find the area of the region that is enclosed by the curves $y = (x-3)/(x^3+x^2)$, $y = 0$, $x = 1$, and $x = 3$.

35–40 Use the Integral Table to evaluate the integral.

35. $\int \sin 7x \cos 10x\,dx$
36. $\int (x^3 - x^2)e^{-x}\,dx$
37. $\int x\sqrt{x - x^2}\,dx$
38. $\int \frac{dx}{x\sqrt{4x+3}}$
39. $\int \tan^2 2x\,dx$
40. $\int \frac{3x-1}{2+x^2}\,dx$

41–42 Approximate the integral using (a) the midpoint approximation M_{10}, (b) the trapezoidal approximation T_{10}, and (c) Simpson's rule approximation S_{20}. In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places.

41. $\int_1^3 \frac{1}{\sqrt{x+1}}\,dx$
42. $\int_{-2}^2 \frac{1}{1+x^2}\,dx$

43–44 Use inequalities (12), (13), and (14) of Section 7.7 to find upper bounds on the errors in parts (a), (b), or (c) of the indicated exercise.

43. Exercise 41
44. Exercise 42

45–46 Use inequalities (12), (13), and (14) of Section 7.7 to find a number n of subintervals for (a) the midpoint approximation M_n, (b) the trapezoidal approximation T_n, and (c) Simpson's rule approximation S_n to ensure the absolute error will be less than 10^{-4}.

45. Exercise 41
46. Exercise 42

47–50 Evaluate the integral if it converges.

47. $\int_0^{+\infty} e^{-x}\,dx$
48. $\int_{-\infty}^2 \frac{dx}{x^2+4}$
49. $\int_0^9 \frac{dx}{\sqrt{9-x}}$
50. $\int_0^1 \frac{1}{2x-1}\,dx$

51. Find the area that is enclosed between the x-axis and the curve $y = (\ln x - 1)/x^2$ for $x \geq e$.

52. Find the volume of the solid that is generated when the region between the x-axis and the curve $y = e^{-x}$ for $x \geq 0$ is revolved about the y-axis.

53. Find a positive value of a that satisfies the equation

$$\int_0^{+\infty} \frac{1}{x^2+a^2}\,dx = 1$$

54. Consider the following methods for evaluating integrals: u-substitution, integration by parts, partial fractions, reduction formulas, and trigonometric substitutions. In each part, state the approach that you would try first to evaluate the integral. If none of them seems appropriate, then say so. You need not evaluate the integral.
   (a) $\int x \sin x\,dx$
   (b) $\int \cos x \sin x\,dx$

(cont.)

(c) $\int \tan^7 x\, dx$ (d) $\int \tan^7 x \sec^2 x\, dx$

(e) $\int \dfrac{3x^2}{x^3+1}\, dx$ (f) $\int \dfrac{3x^2}{(x+1)^3}\, dx$

(g) $\int \tan^{-1} x\, dx$ (h) $\int \sqrt{4-x^2}\, dx$

(i) $\int x\sqrt{4-x^2}\, dx$

55–74 Evaluate the integral.

55. $\displaystyle\int \frac{dx}{(3+x^2)^{3/2}}$

56. $\displaystyle\int x \cos 4x\, dx$

57. $\displaystyle\int_0^{\pi/4} \tan^7 \theta\, d\theta$

58. $\displaystyle\int \frac{\cos\theta}{\sin^2\theta - 6\sin\theta + 12}\, d\theta$

59. $\displaystyle\int \sin^2 2x \cos^3 2x\, dx$

60. $\displaystyle\int_0^4 \frac{1}{(x-2)^2}\, dx$

61. $\displaystyle\int e^{2x} \cos 3x\, dx$

62. $\displaystyle\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx$

63. $\displaystyle\int \frac{dx}{(x-1)(x+2)(x-3)}$

64. $\displaystyle\int_0^{1/3} \frac{dx}{(4-9x^2)^2}$

65. $\displaystyle\int_4^8 \frac{\sqrt{x-4}}{x}\, dx$

66. $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

67. $\displaystyle\int \frac{1}{\sqrt{e^x+1}}\, dx$

68. $\displaystyle\int \frac{dx}{x(x^2+x+1)}$

69. $\displaystyle\int_0^{1/2} \sin^{-1} x\, dx$

70. $\displaystyle\int \tan^5 4x \sec^4 4x\, dx$

71. $\displaystyle\int \frac{x+3}{\sqrt{x^2+2x+2}}\, dx$

72. $\displaystyle\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta}\, d\theta$

73. $\displaystyle\int_a^{+\infty} \frac{x}{(x^2+1)^2}\, dx$

74. $\displaystyle\int_0^{+\infty} \frac{dx}{a^2+b^2x^2}, \quad a, b > 0$

CHAPTER 7 MAKING CONNECTIONS

C CAS

1. Recall from Theorem 6.3.1 and the discussion preceding it that if $f'(x) > 0$, then the function f is increasing and has an inverse function. Parts (a), (b), and (c) of this problem show that if this condition is satisfied and if f' is continuous, then a definite integral of f^{-1} can be expressed in terms of a definite integral of f.
   (a) Use integration by parts to show that
   $$\int_a^b f(x)\, dx = bf(b) - af(a) - \int_a^b xf'(x)\, dx$$
   (b) Use the result in part (a) to show that if $y = f(x)$, then
   $$\int_a^b f(x)\, dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\, dy$$
   (c) Show that if we let $\alpha = f(a)$ and $\beta = f(b)$, then the result in part (b) can be written as
   $$\int_\alpha^\beta f^{-1}(x)\, dx = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\, dx$$

2. In each part, use the result in Exercise 1 to obtain the equation, and then confirm that the equation is correct by performing the integrations.
   (a) $\displaystyle\int_0^{1/2} \sin^{-1} x\, dx = \tfrac{1}{2}\sin^{-1}\left(\tfrac{1}{2}\right) - \int_0^{\pi/6} \sin x\, dx$
   (b) $\displaystyle\int_e^{e^2} \ln x\, dx = (2e^2 - e) - \int_1^2 e^x\, dx$

3. The *Gamma function*, $\Gamma(x)$, is defined as
   $$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt$$
   It can be shown that this improper integral converges if and only if $x > 0$.
   (a) Find $\Gamma(1)$.
   (b) Prove: $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. [*Hint:* Use integration by parts.]
   (c) Use the results in parts (a) and (b) to find $\Gamma(2)$, $\Gamma(3)$, and $\Gamma(4)$; and then make a conjecture about $\Gamma(n)$ for positive integer values of n.
   (d) Show that $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$. [*Hint:* See Exercise 64 of Section 7.8.]
   (e) Use the results obtained in parts (b) and (d) to show that $\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$ and $\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{4}\sqrt{\pi}$.

4. Refer to the Gamma function defined in Exercise 3 to show that
   (a) $\displaystyle\int_0^1 (\ln x)^n\, dx = (-1)^n \Gamma(n+1), \quad n > 0$
   [*Hint:* Let $t = -\ln x$.]
   (b) $\displaystyle\int_0^{+\infty} e^{-x^n}\, dx = \Gamma\left(\frac{n+1}{n}\right), \quad n > 0.$
   [*Hint:* Let $t = x^n$. Use the result in Exercise 3(b).]

C 5. A *simple pendulum* consists of a mass that swings in a vertical plane at the end of a massless rod of length L, as shown in the accompanying figure. Suppose that a simple pendulum is displaced through an angle θ_0 and released from rest. It can be