CHAPTER 6 REVIEW EXERCISES

Graphing Utility

1. In each part, find $f^{-1}(x)$ if the inverse exists.
   (a) $f(x) = (e^x)^2 + 1$
   (b) $f(x) = \sin\left(\dfrac{1-2x}{x}\right), \quad \dfrac{2}{4+\pi} \le x \le \dfrac{2}{4-\pi}$
   (c) $f(x) = \dfrac{1}{1 + 3\tan^{-1} x}$

2. (a) State the restrictions on the domains of $\sin x$, $\cos x$, $\tan x$, and $\sec x$ that are imposed to make those functions one-to-one in the definitions of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, and $\sec^{-1} x$.
   (b) Sketch the graphs of the restricted trigonometric functions in part (a) and their inverses.

3. In each part, find the exact numerical value of the given expression.
   (a) $\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)]$
   (b) $\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)]$

4. In each part, sketch the graph, and check your work with a graphing utility.
   (a) $f(x) = 3\sin^{-1}(x/2)$
   (b) $f(x) = \cos^{-1} x - \pi/2$
   (c) $f(x) = 2\tan^{-1}(-3x)$
   (d) $f(x) = \cos^{-1} x + \sin^{-1} x$

5. Suppose that the graph of $y = \log x$ is drawn with equal scales of 1 inch per unit in both the x- and y-directions. If a bug wants to walk along the graph until it reaches a height of 5 ft above the x-axis, how many miles to the right of the origin will it have to travel?

6. Find the largest value of a such that the function $f(x) = xe^{-x}$ has an inverse on the interval $(-\infty, a]$.

7. Express the following function as a rational function of x:
$$3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1)$$

8. Suppose that $y = Ce^{kt}$, where C and k are constants, and let $Y = \ln y$. Show that the graph of Y versus t is a line, and state its slope and Y-intercept.

9. (a) Sketch the curves $y = \pm e^{-x/2}$ and $y = e^{-x/2}\sin 2x$ for $-\pi/2 \le x \le 3\pi/2$ in the same coordinate system, and check your work using a graphing utility.
   (b) Find all x-intercepts of the curve $y = e^{-x/2}\sin 2x$ in the stated interval, and find the x-coordinates of all points where this curve intersects the curves $y = \pm e^{-x/2}$.

10. Suppose that a package of medical supplies is dropped from a helicopter straight down by parachute into a remote area. The velocity v (in feet per second) of the package t seconds after it is released is given by $v = 24.61(1 - e^{-1.3t})$.
   (a) Graph v versus t.
   (b) Show that the graph has a horizontal asymptote $v = c$.
   (c) The constant c is called the ***terminal velocity***. Explain what the terminal velocity means in practical terms.
   (d) Can the package actually reach its terminal velocity? Explain.
   (e) How long does it take for the package to reach 98% (its terminal velocity?

11. A breeding group of 20 bighorn sheep is released in a prc tected area in Colorado. It is expected that with carefu management the number of sheep, N, after t years will b given by the formula
$$N = \frac{220}{1 + 10(0.83^t)}$$
and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 80.
   (a) Graph N versus t.
   (b) How many years must the state of Colorado maintain a program to care for the sheep?
   (c) How many bighorn sheep can the environment in the protected area support? [*Hint:* Examine the graph of N versus t for large values of t.]

12. An oven is preheated and then remains at a constant temperature. A potato is placed in the oven to bake. Suppose that the temperature T (in °F) of the potato t minutes later is given by $T = 400 - 325(0.97^t)$. The potato will be considered done when its temperature is anywhere between 260°F and 280°F.
   (a) During what interval of time would the potato be considered done?
   (b) How long does it take for the difference between the potato and oven temperatures to be cut in half?

13. (a) Show that the graphs of $y = \ln x$ and $y = x^{0.2}$ intersect.
   (b) Approximate the solution(s) of the equation $\ln x = x^{0.2}$ to three decimal places.

14. (a) Show that for $x > 0$ and $k \ne 0$ the equations
$$x^k = e^x \quad \text{and} \quad \frac{\ln x}{x} = \frac{1}{k}$$
have the same solutions.
   (b) Use the graph of $y = (\ln x)/x$ to determine the values of k for which the equation $x^k = e^x$ has two distinct positive solutions.
   (c) Estimate the positive solution(s) of $x^8 = e^x$.

15–18 Find the limits.

15. $\displaystyle\lim_{t\to\pi/2^+} e^{\tan t}$

16. $\displaystyle\lim_{\theta\to 0^+} \ln(\sin 2\theta) - \ln(\tan\theta)$

17. $\displaystyle\lim_{x\to+\infty} \left(1 + \frac{3}{x}\right)^{-x}$

18. $\displaystyle\lim_{x\to+\infty} \left(1 + \frac{a}{x}\right)^{bx}, \quad a, b > 0$

19–20 Find dy/dx by first using algebraic properties of the natural logarithm function.

19. $y = \ln\left(\dfrac{(x+1)(x+2)^2}{(x+3)^3(x+4)^4}\right)$

20. $y = \ln\left(\dfrac{\sqrt{x}\sqrt[3]{x+1}}{\sin x \sec x}\right)$

21–38 Find dy/dx.

21. $y = \ln 2x$
22. $y = (\ln x)^2$
23. $y = \sqrt[3]{\ln x + 1}$
24. $y = \ln(\sqrt[3]{x+1})$
25. $y = \log(\ln x)$
26. $y = \dfrac{1+\log x}{1-\log x}$
27. $y = \ln(x^{3/2}\sqrt{1+x^4})$
28. $y = \ln\left(\dfrac{\sqrt{x}\cos x}{1+x^2}\right)$
29. $y = e^{\ln(x^2+1)}$
30. $y = \ln\left(\dfrac{1+e^x+e^{2x}}{1-e^{3x}}\right)$
31. $y = 2xe^{\sqrt{x}}$
32. $y = \dfrac{a}{1+be^{-x}}$
33. $y = \dfrac{1}{\pi}\tan^{-1} 2x$
34. $y = 2^{\sin^{-1} x}$
35. $y = x^{(e^x)}$
36. $y = (1+x)^{1/x}$
37. $y = \sec^{-1}(2x+1)$
38. $y = \sqrt{\cos^{-1} x^2}$

39–40 Find dy/dx using logarithmic differentiation.

39. $y = \dfrac{x^3}{\sqrt{x^2+1}}$
40. $y = \sqrt[3]{\dfrac{x^2-1}{x^2+1}}$

41. (a) Make a conjecture about the shape of the graph of $y = \frac{1}{2}x - \ln x$, and draw a rough sketch.
(b) Check your conjecture by graphing the equation over the interval $0 < x < 5$ with a graphing utility.
(c) Show that the slopes of the tangent lines to the curve at $x = 1$ and $x = e$ have opposite signs.
(d) What does part (c) imply about the existence of a horizontal tangent line to the curve? Explain.
(e) Find the exact x-coordinates of all horizontal tangent lines to the curve.

42. Recall from Section 6.1 that the loudness β of a sound in decibels (dB) is given by $\beta = 10\log(I/I_0)$, where I is the intensity of the sound in watts per square meter (W/m^2) and I_0 is a constant that is approximately the intensity of a sound at the threshold of human hearing. Find the rate of change of β with respect to I at the point where
(a) $I/I_0 = 10$ (b) $I/I_0 = 100$ (c) $I/I_0 = 1000$.

43. A particle is moving along the curve $y = x\ln x$. Find all values of x at which the rate of change of y with respect to time is three times that of x. [Assume that dx/dt is never zero.]

44. Find the equation of the tangent line to the graph of $y = \ln(5-x^2)$ at $x = 2$.

45. Find the value of b so that the line $y = x$ is tangent to the graph of $y = \log_b x$. Confirm your result by graphing both $y = x$ and $y = \log_b x$ in the same coordinate system.

46. In each part, find the value of k for which the graphs of $y = f(x)$ and $y = \ln x$ share a common tangent line at their point of intersection. Confirm your result by graphing $y = f(x)$ and $y = \ln x$ in the same coordinate system.
(a) $f(x) = \sqrt{x} + k$ (b) $f(x) = k\sqrt{x}$

47. If f and g are inverse functions and f is differentiable on its domain, must g be differentiable on its domain? Give a reasonable informal argument to support your answer.

48. In each part, find $(f^{-1})'(x)$ using Formula (2) of Section 6.3, and check your answer by differentiating f^{-1} directly.
(a) $f(x) = 3/(x+1)$ (b) $f(x) = \sqrt{e^x}$

49. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

50. Show that the rate of change of $y = 5000e^{1.07x}$ is proportional to y.

51. Show that the function $y = e^{ax}\sin bx$ satisfies
$$y'' - 2ay' + (a^2+b^2)y = 0$$
for any real constants a and b.

52. Show that the function $y = \tan^{-1} x$ satisfies
$$y'' = -2\sin y\cos^3 y$$

53. Suppose that the population of deer on an island is modeled by the equation
$$P(t) = \frac{95}{5-4e^{-t/4}}$$
where $P(t)$ is the number of deer t weeks after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

54. The equilibrium constant k of a balanced chemical reaction changes with the absolute temperature T according to the law
$$k = k_0 \exp\left(-\frac{q(T-T_0)}{2T_0T}\right)$$
where k_0, q, and T_0 are constants. Find the rate of change of k with respect to T.

55–56 Find the limit by interpreting the expression as an appropriate derivative.

55. $\displaystyle\lim_{h\to 0}\frac{(1+h)^\pi - 1}{h}$
56. $\displaystyle\lim_{x\to e}\frac{1-\ln x}{(x-e)\ln x}$

57. Suppose that $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. In each of the four possible cases, state whether $\lim[f(x)-g(x)]$ is an indeterminate form, and give a reasonable informal argument to support your answer.

58. (a) Under what conditions will a limit of the form
$$\lim_{x\to a}[f(x)/g(x)]$$
be an indeterminate form?
(b) If $\lim_{x\to a} g(x) = 0$, must $\lim_{x\to a}[f(x)/g(x)]$ be an indeterminate form? Give some examples to support your answer.

59–62 Evaluate the given limit.

59. $\lim_{x\to+\infty}(e^x - x^2)$

60. $\lim_{x\to 1}\sqrt{\dfrac{\ln x}{x^4-1}}$

61. $\lim_{x\to 0}\dfrac{x^2e^x}{\sin^2 3x}$

62. $\lim_{x\to 0}\dfrac{a^x-1}{x},\quad a>0$

63–64 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

63. $f(x) = 1/e^{x^2}$

64. $f(x) = \tan^{-1} x^2$

65–66 Use any method to find the relative extrema of the function f.

65. $f(x) = \ln(1+x^2)$

66. $f(x) = x^2e^x$

67–68 In each part, find the absolute minimum m and the absolute maximum M of f on the given interval (if they exist), and state where the absolute extrema occur.

67. $f(x) = e^x/x^2;\ (0, +\infty)$

68. $f(x) = x^x;\ (0, +\infty)$

69. Use a graphing utility to estimate the absolute maximum and minimum values of $f(x) = x/2 + \ln(x^2+1)$, if any, on the interval $[-4, 0]$, and then use calculus methods to find the exact values.

70. Prove that $x \le \sin^{-1} x$ for all x in $[0, 1]$.

71–74 Evaluate the integrals.

71. $\int [x^{-2/3} - 5e^x]\,dx$

72. $\int \left[\dfrac{3}{4x} - \sec^2 x\right] dx$

73. $\int \left[\dfrac{1}{1+x^2} + \dfrac{2}{\sqrt{1-x^2}}\right] dx$

74. $\int \left[\dfrac{12}{x\sqrt{x^2-1}} + \dfrac{1-x^4}{1+x^2}\right] dx$

75–76 Use a calculating utility to find the left endpoint, right endpoint, and midpoint approximations to the area under the curve $y = f(x)$ over the stated interval using $n = 10$ subintervals.

75. $y = \ln x;\ [1, 2]$

76. $y = e^x;\ [0, 1]$

77. Interpret the expression as a definite integral over $[0, 1]$, and then evaluate the limit by evaluating the integral.

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} e^{x_k^*}\Delta x_k$$

78. Find the limit

$$\lim_{n\to+\infty}\frac{e^{1/n}+e^{2/n}+e^{3/n}+\cdots+e^{n/n}}{n}$$

by interpreting it as a limit of Riemann sums in which the interval $[0, 1]$ is divided into n subintervals of equal length.

79–80 Find the area under the curve $y = f(x)$ over the stated interval.

79. $f(x) = e^x;\ [1, 3]$

80. $f(x) = \dfrac{1}{x};\ [1, e^3]$

81. Solve the initial-value problems.

(a) $\dfrac{dy}{dx} = \cos x - 5e^x,\ y(0) = 0$

(b) $\dfrac{dy}{dx} = xe^{x^2},\ y(0) = 0$

82–84 Evaluate the integrals by making an appropriate substitution.

82. $\displaystyle\int_e^{e^2}\frac{dx}{x\ln x}$

83. $\displaystyle\int_0^1\frac{dx}{\sqrt{e^x}}$

84. $\displaystyle\int_0^{2/\sqrt{3}}\frac{1}{4+9x^2}\,dx$

85. Find the volume of the solid whose base is the region bounded between the curves $y = \sqrt{x}$ and $y = 1/\sqrt{x}$ for $1 \le x \le 4$ and whose cross sections taken perpendicular to the x-axis are squares.

86. Find the average value of $f(x) = e^x + e^{-x}$ over the interval $\left[\ln\frac{1}{2}, \ln 2\right]$.

87. In each part, prove the identity.

(a) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(b) $\cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x + 1)}$

(c) $\sinh \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(\cosh x - 1)}$

88. Show that for any constant a, the function $y = \sinh(ax)$ satisfies the equation $y'' = a^2y$.

QUICK CHECK EXERCISES 7.2

(See page 500 for answers.)

1. (a) If $G'(x) = g(x)$, then
$$\int f(x)g(x)\,dx = f(x)G(x) - \underline{\qquad}$$
(b) If $u = f(x)$ and $v = G(x)$, then the formula in part (a) can be written in the form $\int u\,dv = \underline{\qquad}$.

2. Find an appropriate choice of u and dv for integration by parts of each integral. Do not evaluate the integral.
(a) $\int x \ln x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(b) $\int (x-2)\sin x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(c) $\int \sin^{-1} x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(d) $\int \dfrac{x}{\sqrt{x-1}}\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$

3. Use integration by parts to evaluate the integral.
(a) $\int xe^{2x}\,dx$ (b) $\int \ln(x-1)\,dx$
(c) $\int_0^{\pi/6} x \sin 3x\,dx$

4. Use a reduction formula to evaluate $\int \sin^3 x\,dx$.

EXERCISE SET 7.2

1–38 Evaluate the integral.

1. $\int xe^{-2x}\,dx$
2. $\int xe^{4x}\,dx$
3. $\int x^2e^{x}\,dx$
4. $\int x^2e^{-2x}\,dx$
5. $\int x^2\cos x\,dx$
6. $\int x\cos 2x\,dx$
7. $\int x\sin 3x\,dx$
8. $\int x^2\sin x\,dx$
9. $\int x\ln x\,dx$
10. $\int \sqrt{x}\ln x\,dx$
11. $\int (\ln x)^2\,dx$
12. $\int \dfrac{\ln x}{\sqrt{x}}\,dx$
13. $\int \ln(3x-2)\,dx$
14. $\int \ln(x^2+16)\,dx$
15. $\int \sin^{-1} 2x\,dx$
16. $\int \cos^{-1}(2x)\,dx$
17. $\int \tan^{-1}(3x)\,dx$
18. $\int x\tan^{-1} x\,dx$
19. $\int \sin(\ln x)\,dx$
20. $\int \cos(\ln x)\,dx$
21. $\int e^{x}\sin x\,dx$
22. $\int e^{3x}\cos 2x\,dx$
23. $\int x\sec^2 x\,dx$
24. $\int x\tan^2 x\,dx$
25. $\int x^3e^{x^2}\,dx$
26. $\int \dfrac{xe^{x}}{(x+1)^2}\,dx$
27. $\int_0^2 xe^{3x}\,dx$
28. $\int_0^1 xe^{-3x}\,dx$
29. $\int_1^e x^2\ln x\,dx$
30. $\int_{\sqrt{e}}^{e} \dfrac{\ln x}{x^2}\,dx$
31. $\int_{-1}^{1} \ln(x+2)\,dx$
32. $\int_0^{\sqrt{3}/2} \sin^{-1} x\,dx$
33. $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$
34. $\int_1^2 x\sec^{-1} x\,dx$
35. $\int_0^{\pi} x\sin 2x\,dx$
36. $\int_0^{\pi} (x + x\cos x)\,dx$
37. $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$
38. $\int_0^2 \ln(x^2+1)\,dx$

39–42 True–False Determine whether the statement is true or false. Explain your answer.

39. The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original.

40. Applying the LIATE strategy to evaluate $\int x^3\ln x\,dx$, we should choose $u = x^3$ and $dv = \ln x\,dx$.

41. To evaluate $\int \sin(\ln x)\,dx$ using integration by parts, choose $dv = \ln\,dx$.

42. Tabular integration by parts is useful for integrals of the form $\int p(x)f(x)\,dx$, where $p(x)$ is a polynomial and $f(x)$ can be repeatedly integrated.

43–44 Evaluate the integral by making a u-substitution and then integrating by parts.

43. $\int \sin\sqrt{x}\,dx$
44. $\int e^{\sqrt{x}}\,dx$

45. Prove that tabular integration by parts gives the correct answer for
$$\int p(x)f(x)\,dx$$
where $p(x)$ is any *quadratic polynomial* and $f(x)$ is any function that can be repeatedly integrated.

46. The computations of any integral evaluated by repeated integration by parts can be organized using tabular integration by parts. Use this organization to evaluate $\int e^{x}\cos x\,dx$ in

two ways: first by repeated differentiation of $\cos x$ (compare Example 5), and then by repeated differentiation of e^x.

47–52 Evaluate the integral using tabular integration by parts.

47. $\int 4x^4 \sin 2x\, dx$

48. $\int (x^2 + x + 1) \sin x\, dx$

49. $\int (3x^2 - x + 2)e^{-x}\, dx$

50. $\int x^3 \sqrt{2x+1}\, dx$

51. $\int e^{ax} \sin bx\, dx$

52. $\int e^{-2\theta} \sin 3\theta\, d\theta$

53. Consider the integral $\int \sin x \cos x\, dx$.
(a) Evaluate the integral two ways: first using integration by parts, and then using the substitution $u = \sin x$.
(b) Show that the results of part (a) are equivalent.
(c) Which of the two methods do you prefer? Discuss the reasons for your preference.

54. Evaluate the integral

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\, dx$$

using
(a) integration by parts
(b) the substitution $u = \sqrt{x^2+1}$.

55. (a) Find the area of the region enclosed by $y = \ln x$, the line $x = e$, and the x-axis.
(b) Find the volume of the solid generated when the region in part (a) is revolved about the x-axis.

56. Find the area of the region between $y = x \sin x$ and $y = x$ for $0 \le x \le \pi/3$.

57. Find the volume of the solid generated when the region between $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ is revolved about the y-axis.

58. Find the volume of the solid generated when the region enclosed between $y = \cos x$ and $y = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

59. A particle moving along the x-axis has velocity function $v(t) = t^2 \sin t$. How far does the particle travel from time $t = 0$ to $t = \pi$?

60. The study of sawtooth waves in electrical engineering leads to integrals of the form

$$\int_{-\pi/\omega}^{\pi/\omega} t \sin(k\omega t)\, dt$$

where k is an integer and ω is a nonzero constant. Evaluate the integral.

61. Use reduction formula (9) to evaluate
(a) $\int \sin^4 x\, dx$ (b) $\int_0^{\pi/2} \sin^5 x\, dx$.

62. Use reduction formula (10) to evaluate
(a) $\int \cos^5 x\, dx$ (b) $\int_0^{\pi/2} \cos^6 x\, dx$.

63. Derive reduction formula (9).

64. In each part, use integration by parts or other methods to derive the reduction formula.

(a) $\displaystyle\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$

(b) $\displaystyle\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$

(c) $\displaystyle\int x^n e^x\, dx = x^n e^x - n\int x^{n-1} e^x\, dx$

65–66 Use the reduction formulas in Exercise 64 to evaluate the integrals.

65. (a) $\int \tan^3 x\, dx$ (b) $\int \sec^4 x\, dx$ (c) $\int x^3 e^x\, dx$

66. (a) $\int x^2 e^{3x}\, dx$ (b) $\int_0^1 x e^{-\sqrt{x}}\, dx$
[*Hint:* First make a substitution.]

67. Let f be a function whose second derivative is continuous on $[-1, 1]$. Show that

$$\int_{-1}^{1} x f''(x)\, dx = f'(1) + f'(-1) - f(1) + f(-1)$$

FOCUS ON CONCEPTS

68. (a) In the integral $\int x \cos x\, dx$, let

$$u = x, \quad dv = \cos x\, dx,$$
$$du = dx, \quad v = \sin x + C_1$$

Show that the constant C_1 cancels out, thus giving the same solution obtained by omitting C_1.
(b) Show that in general

$$uv - \int v\, du = u(v + C_1) - \int (v + C_1)\, du$$

thereby justifying the omission of the constant of integration when calculating v in integration by parts.

69. Evaluate $\int \ln(x+1)\, dx$ using integration by parts. Simplify the computation of $\int v\, du$ by introducing a constant of integration $C_1 = 1$ when going from dv to v.

70. Evaluate $\int \ln(3x-2)\, dx$ using integration by parts. Simplify the computation of $\int v\, du$ by introducing a constant of integration $C_1 = -\frac{2}{3}$ when going from dv to v. Compare your solution with your answer to Exercise 13.

71. Evaluate $\int x \tan^{-1} x\, dx$ using integration by parts. Simplify the computation of $\int v\, du$ by introducing a constant of integration $C_1 = \frac{1}{2}$ when going from dv to v.

72. What equation results if integration by parts is applied to the integral

$$\int \frac{1}{x \ln x}\, dx$$

with the choices

$$u = \frac{1}{\ln x} \quad \text{and} \quad dv = \frac{1}{x}\, dx?$$

In what sense is this equation true? In what sense is it false?