◀ Figure Ex-55

56. Find the average value of $f(x) = x^2 + \dfrac{1}{x^2}$ over the interval $\left[\frac{1}{2}, 2\right]$.

57. Derive the formulas for the position and velocity functions of a particle that moves with constant acceleration along a coordinate line.

58. The velocity of a particle moving along an s-axis is measured at 5 s intervals for 40 s, and the velocity function is modeled by a smooth curve. (The curve and the data points are shown in the accompanying figure.) Use this model in each part.
  (a) Does the particle have constant acceleration? Explain your reasoning.
  (b) Is there any 15 s time interval during which the acceleration is constant? Explain your reasoning.
  (c) Estimate the distance traveled by the particle from time $t = 0$ to time $t = 40$.
  (d) Estimate the average velocity of the particle over the 40 s time period.
  (e) Is the particle ever slowing down during the 40 s time period? Explain your reasoning.
  (f) Is there sufficient information for you to determine the s-coordinate of the particle at time $t = 10$? If so, find it. If not, explain what additional information you need.
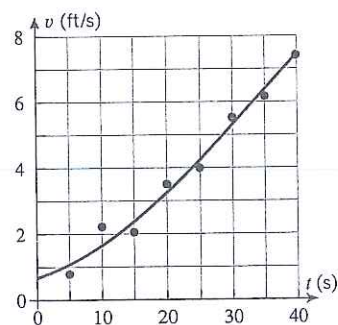

◀ Figure Ex-58

59–62 A particle moves along an s-axis. Use the given information to find the position function of the particle.

59. $v(t) = t^3 - 2t^2 + 1; \ s(0) = 1$

60. $a(t) = 4\cos 2t; \ v(0) = -1, s(0) = -3$

61. $v(t) = 2t - 3; \ s(1) = 5$

62. $a(t) = \cos t - 2t; \ v(0) = 0, s(0) = 0$

63–66 A particle moves with a velocity of $v(t)$ m/s along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval.

63. $v(t) = 2t - 4; \ 0 \le t \le 6$

64. $v(t) = |t - 3|; \ 0 \le t \le 5$

65. $v(t) = \dfrac{1}{2} - \dfrac{1}{t^2}; \ 1 \le t \le 3$

66. $v(t) = \dfrac{3}{\sqrt{t}}; \ 4 \le t \le 9$

67–68 A particle moves with acceleration $a(t)$ m/s^2 along an s-axis and has velocity v_0 m/s at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

67. $a(t) = -2; \ v_0 = 3; \ 1 \le t \le 4$

68. $a(t) = \dfrac{1}{\sqrt{5t + 1}}; \ v_0 = 2; \ 0 \le t \le 3$

69. A car traveling 60 mi/h ($= 88$ ft/s) along a straight road decelerates at a constant rate of 10 ft/s^2.
  (a) How long will it take until the speed is 45 mi/h?
  (b) How far will the car travel before coming to a stop?

70. Suppose that the velocity function of a particle moving along an s-axis is $v(t) = 20t^2 - 100t + 50$ ft/s and that the particle is at the origin at time $t = 0$. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 6 s of motion.

71. A ball is thrown vertically upward from a height of s_0 ft with an initial velocity of v_0 ft/s. If the ball is caught at height s_0, determine its average speed through the air using the free-fall model.

72. A rock, dropped from an unknown height, strikes the ground with a speed of 24 m/s. Find the height from which the rock was dropped.

73–77 Evaluate the integrals by making an appropriate substitution.

73. $\displaystyle\int_0^1 (2x + 1)^4 \, dx$

74. $\displaystyle\int_{-5}^0 x\sqrt{4 - x} \, dx$

75. $\displaystyle\int_0^1 \dfrac{dx}{\sqrt{3x + 1}}$

76. $\displaystyle\int_0^{\sqrt{\pi}} x \sin x^2 \, dx$

77. $\displaystyle\int_0^1 \sin^2(\pi x) \cos(\pi x) \, dx$

78. Find a function f and a number a such that
$$2 + \int_a^x f(t) \, dt = \dfrac{8}{x + 3}$$

## QUICK CHECK EXERCISES 5.1   (See page 355 for answers.)

1. An integral expression for the area of the region between the curves $y = 20 - 3x^2$ and $y = 3\sqrt{x}$ and bounded on the sides by $x = 0$ and $x = 2$ is _____.

2. An integral expression for the area of the parallelogram bounded by $y = 2x + 8$, $y = 2x - 3$, $x = -1$, and $x = 5$ is _____. The value of this integral is _____.

3. (a) The points of intersection for the circle $x^2 + y^2 = 4$ and the line $y = x + 2$ are _____ and _____.

(b) Expressed as a definite integral with respect to x, _____ gives the area of the region inside the circle $x^2 + y^2 = 4$ and above the line $y = x + 2$.

(c) Expressed as a definite integral with respect to y, _____ gives the area of the region described in part (b).

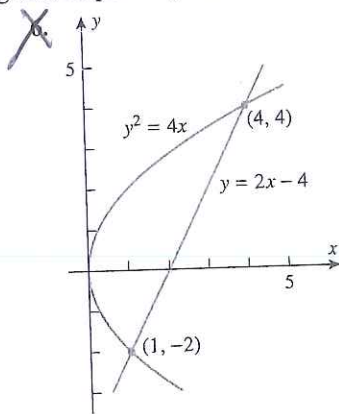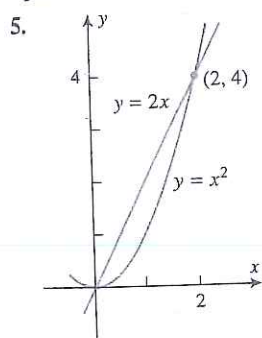4. The area of the region enclosed by the curves $y = x^2$ and $y = \sqrt[3]{x}$ is _____.

EXERCISE SET 5.1   ⌁ Graphing Utility   [c] CAS

1-4 Find the area of the shaded region.

1.

$y = x^2 + 1$
$y = x$

2.

$y = \sqrt{x}$
$y = -\frac{1}{4}x$

3.

$x = y$
$x = 1/y^2$

4.

$x = -y$
$x = 2 - y^2$
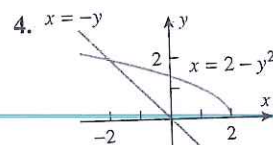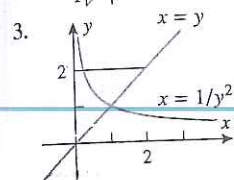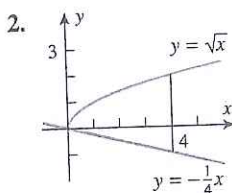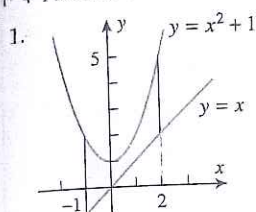
5-6 Find the area of the shaded region by (a) integrating with respect to x and (b) integrating with respect to y.

5.

$(2, 4)$
$y = 2x$
$y = x^2$

6.

$y^2 = 4x$
$(4, 4)$
$y = 2x - 4$
$(1, -2)$

7-14 Sketch the region enclosed by the curves and find its area.

7. $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$

8. $y = x^3 - 4x$, $y = 0$, $x = 0$, $x = 2$

9. $y = \cos 2x$, $y = 0$, $x = \pi/4$, $x = \pi/2$

10. $y = \sec^2 x$, $y = 2$, $x = -\pi/4$, $x = \pi/4$

11. $x = \sin y$, $x = 0$, $y = \pi/4$, $y = 3\pi/4$

12. $x^2 = y$, $x = y - 2$

13. $y = 2 + |x - 1|$, $y = -\frac{1}{5}x + 7$

14. $y = x$, $y = 4x$, $y = -x + 2$

⌁ 15-20 Use a graphing utility, where helpful, to find the area of the region enclosed by the curves.

15. $y = x^3 - 4x^2 + 3x$, $y = 0$

16. $y = x^3 - 2x^2$, $y = 2x^2 - 3x$

17. $y = \sin x$, $y = \cos x$, $x = 0$, $x = 2\pi$

18. $y = x^3 - 4x$, $y = 0$

19. $x = y^3 - y$, $x = 0$

20. $x = y^3 - 4y^2 + 3y$, $x = y^2 - y$

21-24 True-False Determine whether the statement is true or false. Explain your answer. [In each exercise, assume that f and g are distinct continuous functions on $[a, b]$ and that A denotes the area of the region bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.]

21. If f and g differ by a positive constant c, then $A = c(b - a)$.

22. If
$$\int_a^b [f(x) - g(x)]\, dx = -3$$
then $A = 3$.

23. If
$$\int_a^b [f(x) - g(x)]\, dx = 0$$
then the graphs of $y = f(x)$ and $y = g(x)$ cross at least once on $[a, b]$.

24. If
$$A = \left| \int_a^b [f(x) - g(x)]\, dx \right|$$
then the graphs of $y = f(x)$ and $y = g(x)$ don't cross on $[a, b]$.

[c] 25. Use a CAS to find the area enclosed by $y = 3 - 2x$ and $y = x^6 + 2x^5 - 3x^4 + x^2$.

[c] 26. Use a CAS to find the exact area enclosed by the curves $y = x^5 - 2x^3 - 3x$ and $y = x^3$.

27. Find a horizontal line $y = k$ that divides the area between $y = x^2$ and $y = 9$ into two equal parts.
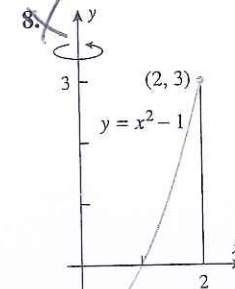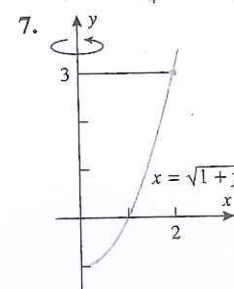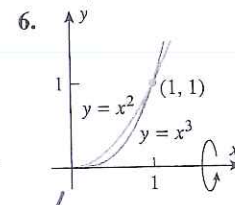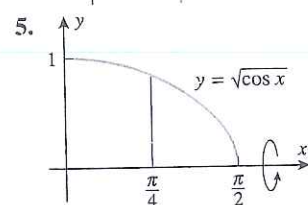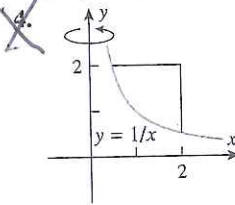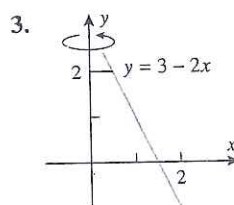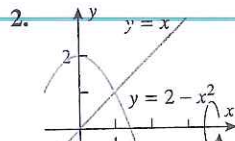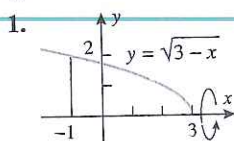
28. Find a vertical line $x = k$ that divides the area enclosed by $x = \sqrt{y}$, $x = 2$, and $y = 0$ into two equal parts.

✔ QUICK CHECK EXERCISES 5.2   (See page 365 for answers.)

1. A solid S extends along the x-axis from $x = 1$ to $x = 3$. For x between 1 and 3, the cross-sectional area of S perpendicular to the x-axis is $3x^2$. An integral expression for the volume of S is _____. The value of this integral is _____.

2. A solid S is generated by revolving the region between the x-axis and the curve $y = \sqrt{\sin x}$ $(0 \le x \le \pi)$ about the x-axis.
   (a) For x between 0 and π, the cross-sectional area of S perpendicular to the x-axis at x is $A(x) =$ _____.
   (b) An integral expression for the volume of S is _____.
   (c) The value of the integral in part (b) is _____.

3. A solid S is generated by revolving the region enclosed by the line $y = 2x + 1$ and the curve $y = x^2 + 1$ about the x-axis.
   (a) For x between _____ and _____, the cross-sectional area of S perpendicular to the x-axis at x is $A(x) =$ _____.
   (b) An integral expression for the volume of S is _____.

4. A solid S is generated by revolving the region enclosed by the line $y = x + 1$ and the curve $y = x^2 + 1$ about the y-axis.
   (a) For y between _____ and _____, the cross-sectional area of S perpendicular to the y-axis at y is $A(y) =$ _____.
   (b) An integral expression for the volume of S is _____.

EXERCISE SET 5.2   [c] CAS

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis.

1.

$y = \sqrt{3 - x}$

2.

$y = x$
$y = 2 - x^2$

3.

$y = 3 - 2x$

4.

$y = 1/x$

5.

$y = \sqrt{\cos x}$

6.

$y = x^2$ (1, 1)
$y = x^3$

7.

$x = \sqrt{1 + y}$

8.

(2, 3)
$y = x^2 - 1$

9. Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the x-axis are squares.

10. Find the volume of the solid whose base is the region bounded between the curve $y = \sec x$ and the x-axis from $x = \pi/4$ to $x = \pi/3$ and whose cross sections taken perpendicular to the x-axis are squares.

11–14 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

11. $y = \sqrt{25 - x^2}$, $y = 3$

12. $y = 9 - x^2$, $y = 0$ 13. $x = \sqrt{y}$, $x = y/4$

14. $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$
    [Hint: Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

15. Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.

16. Find the volume of the solid whose base is the region enclosed between the curve $x = 1 - y^2$ and the y-axis and whose cross sections taken perpendicular to the y-axis are squares.

17–20 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis.

17. $x = \csc y$, $y = \pi/4$, $y = 3\pi/4$, $x = 0$

18. $y = x^2$, $x = y^2$

19. $x = y^2$, $x = y + 2$

20. $x = 1 - y^2$, $x = 2 + y^2$, $y = -1$, $y = 1$

21-24 True-False Determine whether the statement is true or false. Explain your answer. [In these exercises, assume that a solid S of volume V is bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$ and that for each x in $[a, b]$, $A(x)$ denotes the cross-sectional area of S perpendicular to the x-axis.] ■

21. If each cross section of S perpendicular to the x-axis is a square, then S is a rectangular parallelepiped (i.e., is box shaped).

22. If each cross section of S is a disk or a washer, then S is a solid of revolution.

23. If x is in centimeters (cm), then $A(x)$ must be a quadratic function of x, since units of $A(x)$ will be square centimeters (cm^2).

24. The average value of $A(x)$ on the interval $[a, b]$ is given by $V/(b - a)$.

25. Find the volume of the solid that results when the region above the x-axis and below the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

is revolved about the x-axis.

26. Let V be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ $(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which $V = 3$.

27. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x + 1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

28. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

FOCUS ON CONCEPTS

29. Suppose that f is a continuous function on $[a, b]$, and let R be the region between the curve $y = f(x)$ and the line $y = k$ from $x = a$ to $x = b$. Using the method of disks, derive with explanation a formula for the volume of a solid generated by revolving R about the line $y = k$. State and explain additional assumptions, if any, that you need about f for your formula.

30. Suppose that v and w are continuous functions on $[c, d]$, and let R be the region between the curves $x = v(y)$ and $x = w(y)$ from $y = c$ to $y = d$. Using the method of washers, derive with explanation a formula for the volume of a solid generated by revolving R about the line $x = k$. State and explain additional assumptions, if any, that you need about v and w for your formula.

31. Consider the solid generated by revolving the shaded region in Exercise 1 about the line $y = 2$.

(a) Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 1. Explain the basis of your conjecture.

(b) Check your conjecture by calculating this volume and comparing it to the volume obtained in Exercise 1.

32. Consider the solid generated by revolving the shaded region in Exercise 4 about the line $x = 2.5$.

(a) Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 4. Explain the basis of your conjecture.

(b) Check your conjecture by expressing the difference in the two volumes as a single definite integral. [*Hint:* Sketch the graph of the integrand.]

33. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $x = 9$.

34. Find the volume of the solid that results when the region in Exercise 33 is revolved about the line $y = 3$.

35. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $y = -1$.

36. Find the volume of the solid that results when the region in Exercise 35 is revolved about the line $x = -1$.

37. Find the volume of the solid that results when the region enclosed by $y = x^2$ and $y = x^3$ is revolved about the line $x = 1$.

38. Find the volume of the solid that results when the region in Exercise 37 is revolved about the line $y = -1$.

39. A nose cone for a space reentry vehicle is designed so that a cross section, taken x ft from the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4}x^2$ ft. Find the volume of the nose cone given that its length is 20 ft.

40. A certain solid is 1 ft high, and a horizontal cross section taken x ft above the bottom of the solid is an annulus of inner radius x^2 ft and outer radius $\sqrt{x}$ ft. Find the volume of the solid.

41. Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the x-axis are squares.

42. The base of a certain solid is the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Every cross section perpendicular to the x-axis is a semicircle with its diameter across the base. Find the volume of the solid.

43. In parts (a)-(c) find the volume of the solid whose base is enclosed by the circle $x^2 + y^2 = 1$ and whose cross sections taken perpendicular to the x-axis are as indicated. *(cont.)*

EXERCISE SET 5.3   ☐ CAS

1–4 Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis. ■

1.

2.

3.

4.
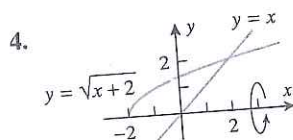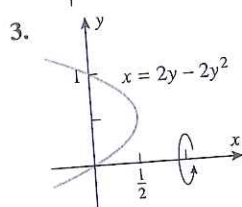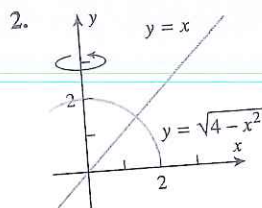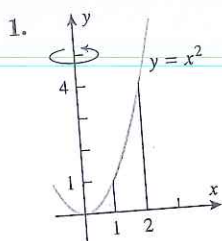
5–10 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis. ■

5. $y = x^3$, $x = 1$, $y = 0$

6. $y = \sqrt{x}$, $x = 4$, $x = 9$, $y = 0$

7. $y = 1/x$, $y = 0$, $x = 1$, $x = 3$

8. $y = \cos(x^2)$, $x = 0$, $x = \frac{1}{2}\sqrt{\pi}$, $y = 0$

9. $y = 2x - 1$, $y = -2x + 3$, $x = 2$

10. $y = 2x - x^2$, $y = 0$

11–14 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the x-axis. ■

11. $y^2 = x$, $y = 1$, $x = 0$

12. $x = 2y$, $y = 2$, $y = 3$, $x = 0$

13. $y = x^2$, $x = 1$, $y = 0$ 14. $xy = 4$, $x + y = 5$

15–18 True–False Determine whether the statement is true or false. Explain your answer. ■

15. The volume of a cylindrical shell is equal to the product of the thickness of the shell with the surface area of a cylinder whose height is that of the shell and whose radius is equal to the average of the inner and outer radii of the shell.

16. The method of cylindrical shells is a special case of the method of integration of cross-sectional area that was discussed in Section 5.2.

17. In the method of cylindrical shells, integration is over an interval on a coordinate axis that is *perpendicular* to the axis of revolution of the solid.

18. The Riemann sum approximation

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x_k \quad \left(\text{where } x_k^* = \frac{x_k + x_{k-1}}{2} \right)$$

for the volume of a solid of revolution is exact when f is a constant function.

☐ 19. Use a CAS to find the volume of the solid generated when the region enclosed by $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ is revolved about the y-axis.

☐ 20. Use a CAS to find the volume of the solid generated when the region enclosed by $y = \cos x$, $y = 0$, and $x = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

☐ 21. Consider the region to the right of the y-axis, to the left of the vertical line $x = k$ $(0 < k < \pi)$, and between the curve $y = \sin x$ and the x-axis. Use a CAS to estimate the value of k so that the solid generated by revolving the region about the y-axis has a volume of 8 cubic units.

FOCUS ON CONCEPTS

22. Let R_1 and R_2 be regions of the form shown in the accompanying figure. Use cylindrical shells to find a formula for the volume of the solid that results when
  (a) region R_1 is revolved about the y-axis
 (b) region R_2 is revolved about the x-axis.
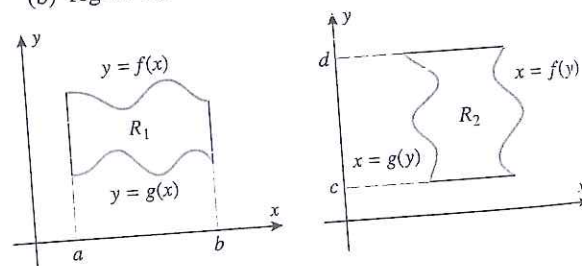
▲ Figure Ex-22

23. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve
$$y = x^3 - 3x^2 + 2x$$
over $[0, 1]$ is revolved about the y-axis.
  (b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.

24. Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is enclosed by $y = f(x)$ and $y = 0$ for $a \le x \le b$. Using the method of cylindrical shells, derive with explanation a formula for the volume of the solid generated by revolving R about the line $x = k$, where $k \le a$.

25–26 Using the method of cylindrical shells, set up but do not evaluate an integral for the volume of the solid generated when the region R is revolved about (a) the line $x = 1$ and (b) the line $y = -1$. ■

25. R is the region bounded by the graphs of $y = x$, $y = 0$, and $x = 1$.

26. R is the region in the first quadrant bounded by the graphs of $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$.

27. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = 1/x^3$, $x = 1$, $x = 2$, $y = 0$ is revolved about the line $x = -1$.

28. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

29. Use cylindrical shells to find the volume of the cone generated when the triangle with vertices $(0, 0)$, $(0, r)$, $(h, 0)$, where $r > 0$ and $h > 0$, is revolved about the x-axis.

30. The region enclosed between the curve $y^2 = kx$ and the line $x = \frac{1}{4}k$ is revolved about the line $x = \frac{1}{2}k$. Use cylindrical shells to find the volume of the resulting solid. (Assume $k > 0$.)

31. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the center of a sphere. Show that the volume of the remaining solid depends only on the length L of the hole, not on the size of the sphere.
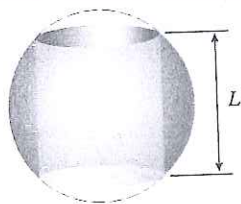
32. Use cylindrical shells to find the volume of the torus obtained by revolving the circle $x^2 + y^2 = a^2$ about the line $x = b$, where $b > a > 0$. [*Hint:* It may help in the integration to think of an integral as an area.]

33. Let V_x and V_y be the volumes of the solids that result when the region enclosed by $y = 1/x$, $y = 0$, $x = \frac{1}{2}$, and $x = b$ $\left(b > \frac{1}{2}\right)$ is revolved about the x-axis and y-axis, respectively. Is there a value of b for which $V_x = V_y$?

34. **Writing** Faced with the problem of computing the volume of a solid of revolution, how would you go about deciding whether to use the method of disks/washers or the method of cylindrical shells?

35. **Writing** With both the method of disks/washers and with the method of cylindrical shells, we integrate an "area" to get the volume of a solid of revolution. However, these two approaches differ in very significant ways. Write a brief paragraph that discusses these differences.

◀ Figure Ex-31

✔ QUICK CHECK ANSWERS 5.3

1. (a) $2\pi x(1 + \sqrt{x})$ (b) $\int_1^4 2\pi x(1 + \sqrt{x})\,dx$ 2. (a) $2\pi(5 - x)(1 + \sqrt{x})$ (b) $\int_1^4 2\pi(5 - x)(1 + \sqrt{x})\,dx$

3. $\int_0^4 2\pi y[4 - (y - 2)^2]\,dy$

5.4 LENGTH OF A PLANE CURVE

In this section we will use the tools of calculus to study the problem of finding the length of a plane curve.
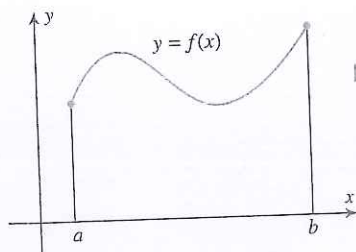
▲ Figure 5.4.1

■ ARC LENGTH

Our first objective is to define what we mean by the **length** (also called the **arc length**) of a plane curve $y = f(x)$ over an interval $[a, b]$ (Figure 5.4.1). Once that is done we will be able to focus on the problem of computing arc lengths. To avoid some complications that would otherwise occur, we will impose the requirement that f' be continuous on $[a, b]$, in which case we will say that $y = f(x)$ is a **smooth curve** on $[a, b]$ or that f is a **smooth function** on $[a, b]$. Thus, we will be concerned with the following problem.

3–8 Find the exact arc length of the curve over the interval.

3. $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$

4. $x = \frac{1}{3}(y^2 + 2)^{3/2}$ from $y = 0$ to $y = 1$

5. $y = x^{2/3}$ from $x = 1$ to $x = 8$

6. $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$

7. $24xy = y^4 + 48$ from $y = 2$ to $y = 4$

8. $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y = 1$ to $y = 4$

9–12 True–False Determine whether the statement is true or false. Explain your answer.

9. The graph of $y = \sqrt{1 - x^2}$ is a smooth curve on $[-1, 1]$.

10. The approximation

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

for arc length is not expressed in the form of a Riemann sum.

11. The approximation

$$L \approx \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k$$

for arc length is exact when f is a linear function of x.

12. In our definition of the arc length for the graph of $y = f(x)$, we need $f'(x)$ to be a continuous function in order for f to satisfy the hypotheses of the Mean-Value Theorem (3.8.2).

FOCUS ON CONCEPTS

13. Consider the curve $y = x^{2/3}$.
    (a) Sketch the portion of the curve between $x = -1$ and $x = 8$.
    (b) Explain why Formula (4) cannot be used to find the arc length of the curve sketched in part (a).
    (c) Find the arc length of the curve sketched in part (a).

14. The curve segment $y = x^2$ from $x = 1$ to $x = 2$ may also be expressed as the graph of $x = \sqrt{y}$ from $y = 1$ to $y = 4$. Set up two integrals that give the arc length of this curve segment, one by integrating with respect to x, and the other by integrating with respect to y. Demonstrate a substitution that verifies that these two integrals are equal.

15. Consider the curve segments $y = x^2$ from $x = \frac{1}{2}$ to $x = 2$ and $y = \sqrt{x}$ from $x = \frac{1}{4}$ to $x = 4$.
    (a) Graph the two curve segments and use your graphs to explain why the lengths of these two curve segments should be equal.
    (b) Set up integrals that give the arc lengths of the curve segments by integrating with respect to x. Demonstrate a substitution that verifies that these two integrals are equal.
    (c) Set up integrals that give the arc lengths of the curve segments by integrating with respect to y.

(d) Approximate the arc length of each curve segment using Formula (2) with $n = 10$ equal subintervals.

(e) Which of the two approximations in part (d) is more accurate? Explain.

(f) Use the midpoint approximation with $n = 10$ subintervals to approximate each arc length integral in part (b).

(g) Use a calculating utility with numerical integration capabilities to approximate the arc length integrals in part (b) to four decimal places.

16. Follow the directions of Exercise 15 for the curve segments $y = x^{8/3}$ from $x = 10^{-3}$ to $x = 1$ and $y = x^{3/8}$ from $x = 10^{-8}$ to $x = 1$.

17. Follow the directions of Exercise 15 for the curve segment $y = 1 + 1/x$ from $x = 1$ to $x = 3$ and for the curve segment $y = 1/(x - 1)$ from $x = 4/3$ to $x = 2$.

18. Let $y = f(x)$ be a smooth curve on the closed interval $[a, b]$. Prove that if m and M are nonnegative numbers such that $m \leq |f'(x)| \leq M$ for all x in $[a, b]$, then the arc length L of $y = f(x)$ over the interval $[a, b]$ satisfies the inequalities

$$(b - a)\sqrt{1 + m^2} \leq L \leq (b - a)\sqrt{1 + M^2}$$

19. Use the result of Exercise 18 to show that the arc length L of $y = \sec x$ over the interval $0 \leq x \leq \pi/3$ satisfies

$$\frac{\pi}{3} \leq L \leq \frac{\pi}{3}\sqrt{13}$$

20. A basketball player makes a successful shot from the free throw line. Suppose that the path of the ball from the moment of release to the moment it enters the hoop is described by

$$y = 2.15 + 2.09x - 0.41x^2, \quad 0 \leq x \leq 4.6$$

where x is the horizontal distance (in meters) from the point of release, and y is the vertical distance (in meters) above the floor. Use a CAS or a scientific calculator with a numerical integration capability to approximate the distance the ball travels from the moment it is released to the moment it enters the hoop. Round your answer to two decimal places.

21. The central span of the Golden Gate Bridge in California is 4200 ft long and is suspended from cables that rise 500 ft above the roadway on either side. Approximately how long is the portion of a cable that lies between the support towers on one side of the roadway? [*Hint:* As suggested by the accompanying figure on the next page, assume the cable is modeled by a parabola $y = ax^2$ that passes through the point $(2100, 500)$. Use a CAS or a calculating utility with a numerical integration capability to approximate the length of the cable. Round your answer to the nearest foot.]
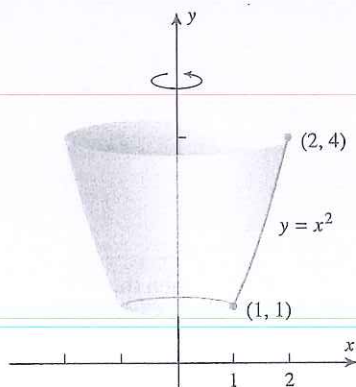
▶ **Example 2** Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.

Solution. First sketch the curve; then imagine revolving it about the y-axis (Figure 5.5.7). Because the curve is revolved about the y-axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y-values corresponding to $x = 1$ and $x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

$$S = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^4 2\pi \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= \pi \int_1^4 \sqrt{4y + 1}\, dy$$

$$= \frac{\pi}{4} \int_5^{17} u^{1/2}\, du \qquad \begin{array}{l} u = 4y + 1 \\ du = 4\, dy \end{array}$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big]_{u=5}^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85 \quad ◀$$

(2, 4)

$y = x^2$

(1, 1)

1   2

▲ **Figure 5.5.7**

✔ **QUICK CHECK EXERCISES 5.5**   *(See page 382 for answers.)*

1. If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is _____.

2. The lateral area of the frustum with slant height $\sqrt{10}$ and base radii $r_1 = 1$ and $r_2 = 2$ is _____.

3. An integral expression for the area of the surface generated by rotating the line segment joining $(3, 1)$ and $(6, 2)$ about the x-axis is _____.

4. An integral expression for the area of the surface generated by rotating the line segment joining $(3, 1)$ and $(6, 2)$ about the y-axis is _____.

EXERCISE SET 5.5   C CAS

1–4 Find the area of the surface generated by revolving the given curve about the x-axis.

1. $y = 7x, \ 0 \leq x \leq 1$

2. $y = \sqrt{x}, \ 1 \leq x \leq 4$

3. $y = \sqrt{4 - x^2}, \ -1 \leq x \leq 1$

4. $x = \sqrt[3]{y}, \ 1 \leq y \leq 8$

5–8 Find the area of the surface generated by revolving the given curve about the y-axis.

5. $x = 9y + 1, \ 0 \leq y \leq 2$

6. $x = y^3, \ 0 \leq y \leq 1$

7. $x = \sqrt{9 - y^2}, \ -2 \leq y \leq 2$

8. $x = 2\sqrt{1 - y}, \ -1 \leq y \leq 0$

C **9–12** Use a CAS to find the exact area of the surface generated by revolving the curve about the stated axis.

9. $y = \sqrt{x} - \frac{1}{3}x^{3/2}, \ 1 \leq x \leq 3; \ x$-axis

10. $y = \frac{1}{3}x^3 + \frac{1}{4}x^{-1}, \ 1 \leq x \leq 2; \ x$-axis

11. $8xy^2 = 2y^6 + 1, \ 1 \leq y \leq 2; \ y$-axis

12. $x = \sqrt{16 - y}, \ 0 \leq y \leq 15; \ y$-axis

C **13–14** Use a CAS or a calculating utility with a numerical integration capability to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places.

13. $y = \sin x, \ 0 \leq x \leq \pi; \ x$-axis

14. $x = \tan y, \ 0 \leq y \leq \pi/4; \ y$-axis