✓ QUICK CHECK EXERCISES 5.1

(See page 355 for answers.)

1. An integral expression for the area of the region between the curves $y = 20 - 3x^2$ and $y = 3\sqrt{x}$ and bounded on the sides by $x = 0$ and $x = 2$ is ______.

2. An integral expression for the area of the parallelogram bounded by $y = 2x + 8$, $y = 2x - 3$, $x = -1$, and $x = 5$ is ______. The value of this integral is ______.

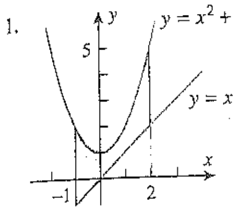
3. (a) The points of intersection for the circle $x^2 + y^2 = 4$ and the line $y = x + 2$ are ______ and ______.
   (b) Expressed as a definite integral with respect to x, ______ gives the area of the region inside the circle $x^2 + y^2 = 4$ and above the line $y = x + 2$.
   (c) Expressed as a definite integral with respect to y, ______ gives the area of the region described in part (b).

4. The area of the region enclosed by the curves $y = x^2$ and $y = \sqrt[3]{x}$ is ______.

EXERCISE SET 5.1

Graphing Utility   C CAS

1–4 Find the area of the shaded region.

1.


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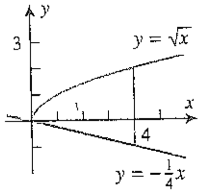

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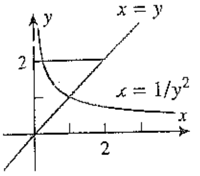

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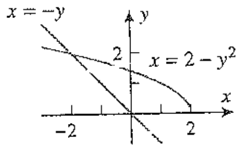
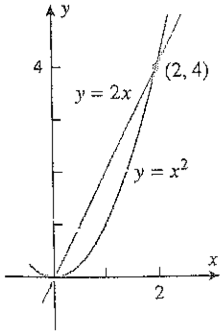


5–6 Find the area of the shaded region by (a) integrating with respect to x and (b) integrating with respect to y.

5.


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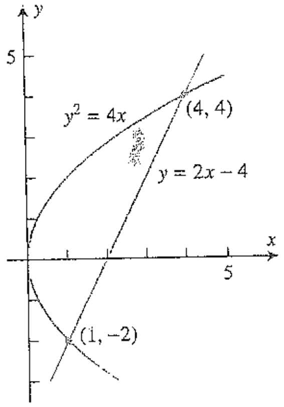


7–14 Sketch the region enclosed by the curves and find its area.

7. $y = x^2,\ y = \sqrt{x},\ x = \frac{1}{4},\ x = 1$
8. $y = x^3 - 4x,\ y = 0,\ x = 0,\ x = 2$
9. $y = \cos 2x,\ y = 0,\ x = \pi/4,\ x = \pi/2$
10. $y = \sec^2 x,\ y = 2,\ x = -\pi/4,\ x = \pi/4$
11. $x = \sin y,\ x = 0,\ y = \pi/4,\ y = 3\pi/4$
12. $x^2 = y,\ x = y - 2$
13. $y = 2 + |x - 1|,\ y = -\frac{1}{5}x + 7$
14. $y = x,\ y = 4x,\ y = -x + 2$

15–20 Use a graphing utility, where helpful, to find the area of the region enclosed by the curves.

15. $y = x^3 - 4x^2 + 3x,\ y = 0$
16. $y = x^3 - 2x^2,\ y = 2x^2 - 3x$
17. $y = \sin x,\ y = \cos x,\ x = 0,\ x = 2\pi$
18. $y = x^3 - 4x,\ y = 0$
19. $x = y^3 - y,\ x = 0$
20. $x = y^3 - 4y^2 + 3y,\ x = y^2 - y$

21–24 True–False Determine whether the statement is true or false. Explain your answer. [In each exercise, assume that f and g are distinct continuous functions on $[a, b]$ and that A denotes the area of the region bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.]

21. If f and g differ by a positive constant c, then $A = c(b - a)$.

22. If
$$\int_a^b [f(x) - g(x)]\,dx = -3$$
then $A = 3$.

23. If
$$\int_a^b [f(x) - g(x)]\,dx = 0$$
then the graphs of $y = f(x)$ and $y = g(x)$ cross at least once on $[a, b]$.

24. If
$$A = \left|\int_a^b [f(x) - g(x)]\,dx\right|$$
then the graphs of $y = f(x)$ and $y = g(x)$ don't cross on $[a, b]$.

25. C Use a CAS to find the area enclosed by $y = 3 - 2x$ and $y = x^6 + 2x^5 - 3x^4 + x^2$.

26. C Use a CAS to find the exact area enclosed by the curves $y = x^5 - 2x^3 - 3x$ and $y = x^3$.

27. Find a horizontal line $y = k$ that divides the area between $y = x^2$ and $y = 9$ into two equal parts.

28. Find a vertical line $x = k$ that divides the area enclosed by $x = \sqrt{y}$, $x = 2$, and $y = 0$ into two equal parts.

29. (a) Find the area of the region enclosed by the parabola $y = 2x - x^2$ and the x-axis.
(b) Find the value of m so that the line $y = mx$ divides the region in part (a) into two regions of equal area.

30. Find the area between the curve $y = \sin x$ and the line segment joining the points $(0, 0)$ and $(5\pi/6, 1/2)$ on the curve.

31–33 Use Newton's Method (Section 3.7), where needed, to approximate the x-coordinates of the intersections of the curves to at least four decimal places, and then use those approximations to approximate the area of the region.

31. The region that lies below the curve $y = \sin x$ and above the line $y = 0.2x$, where $x \geq 0$.

32. The region enclosed by the graphs of $y = x^2$ and $y = \cos x$.

33. The region that is enclosed by the curves $y = x^2 - 1$ and $y = 2 \sin x$.

C 34. Referring to the accompanying figure, use a CAS to estimate the value of k so that the areas of the shaded regions are equal.
Source: This exercise is based on Problem A1 that was posed in the Fifty-Fourth Annual William Lowell Putnam Mathematical Competition.

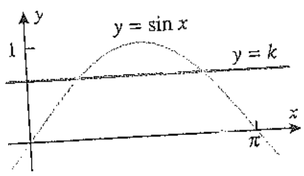


Figure Ex-34

FOCUS ON CONCEPTS

35. Two racers in adjacent lanes move with velocity functions $v_1(t)$ m/s and $v_2(t)$ m/s, respectively. Suppose that the racers are even at time $t = 60$ s. Interpret the value of the integral
$$\int_0^{60} [v_2(t) - v_1(t)]\, dt$$
in this context.

36. The accompanying figure shows acceleration versus time curves for two cars that move along a straight track, accelerating from rest at the starting line. What does the area A between the curves over the interval $0 \leq t \leq T$ represent? Justify your answer.

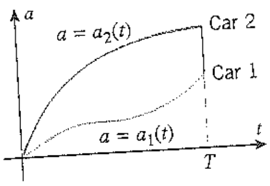


Figure Ex-36

37. The curves in the accompanying figure model the birth rates and death rates (in millions of people per year) for a country over a 50-year period. What does the area A between the curves over the interval [1960, 2010] represent? Justify your answer.

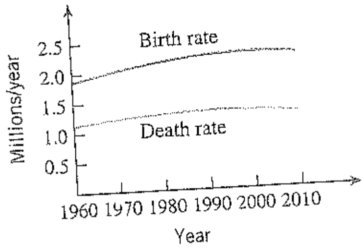


Figure Ex-37

38. The accompanying figure shows the rate at which transdermal medication is absorbed into the bloodstream of an individual, as well as the rate at which the medication is eliminated from the bloodstream by metabolization. Both rates are in units of micrograms per hour (μg/h) and are displayed over an 8-hour period. What does the area A between the curves over the interval [0, 8] represent? Justify your answer.

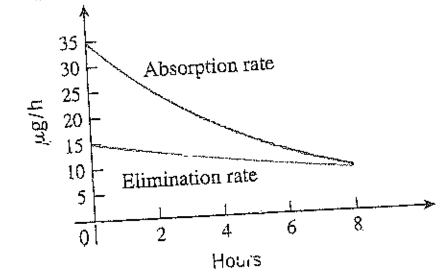


Figure Ex-38

39. Find the area of the region enclosed between the curve $x^{1/2} + y^{1/2} = a^{1/2}$ and the coordinate axes.

40. Show that the area of the ellipse in the accompanying figure is πab. [*Hint:* Use a formula from geometry.]

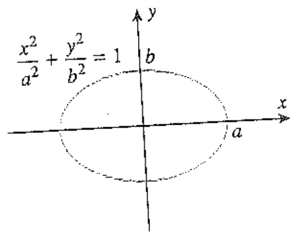


Figure Ex-40

41. **Writing** Suppose that f and g are continuous on $[a, b]$ but that the graphs of $y = f(x)$ and $y = g(x)$ cross several times. Describe a step-by-step procedure for determining the area bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.

42. **Writing** Suppose that R and S are two regions in the xy-plane that lie between a pair of lines L_1 and L_2 that are parallel to the y-axis. Assume that each line between L_1 and L_2 that is parallel to the y-axis intersects R and S in line segments of equal length. Give an informal argument that the area of R is equal to the area of S. (Make reasonable assumptions about the boundaries of R and S.)

21–24 True–False Determine whether the statement is true or false. Explain your answer. [In these exercises, assume that a solid S of volume V is bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$ and that for each x in $[a, b]$, $A(x)$ denotes the cross-sectional area of S perpendicular to the x-axis.]

21. If each cross section of S perpendicular to the x-axis is a square, then S is a rectangular parallelepiped (i.e., is box shaped).

22. If each cross section of S is a disk or a washer, then S is a solid of revolution.

23. If x is in centimeters (cm), then $A(x)$ must be a quadratic function of x, since units of $A(x)$ will be square centimeters (cm^2).

24. The average value of $A(x)$ on the interval $[a, b]$ is given by $V/(b-a)$.

25. Find the volume of the solid that results when the region above the x-axis and below the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

is revolved about the x-axis.

26. Let V be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ $(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which $V = 3$.

27. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x+1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

28. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

FOCUS ON CONCEPTS

29. Suppose that f is a continuous function on $[a, b]$, and let R be the region between the curve $y = f(x)$ and the line $y = k$ from $x = a$ to $x = b$. Using the method of disks, derive with explanation a formula for the volume of a solid generated by revolving R about the line $y = k$. State and explain additional assumptions, if any, that you need about f for your formula.

30. Suppose that v and w are continuous functions on $[c, d]$, and let R be the region between the curves $x = v(y)$ and $x = w(y)$ from $y = c$ to $y = d$. Using the method of washers, derive with explanation a formula for the volume of a solid generated by revolving R about the line $x = k$. State and explain additional assumptions, if any, that you need about v and w for your formula.

31. Consider the solid generated by revolving the shaded region in Exercise 1 about the line $y = 2$.
    (a) Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 1. Explain the basis of your conjecture.
    (b) Check your conjecture by calculating this volume and comparing it to the volume obtained in Exercise 1.

32. Consider the solid generated by revolving the shaded region in Exercise 4 about the line $x = 2.5$.
    (a) Make a conjecture as to which is larger: the volume of this solid or the volume of the solid in Exercise 4. Explain the basis of your conjecture.
    (b) Check your conjecture by expressing the difference in the two volumes as a single definite integral. [*Hint:* Sketch the graph of the integrand.]

33. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $x = 9$.

34. Find the volume of the solid that results when the region in Exercise 33 is revolved about the line $y = 3$.

35. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $y = -1$.

36. Find the volume of the solid that results when the region in Exercise 35 is revolved about the line $x = -1$.

37. Find the volume of the solid that results when the region enclosed by $y = x^2$ and $y = x^3$ is revolved about the line $x = 1$.

38. Find the volume of the solid that results when the region in Exercise 37 is revolved about the line $y = -1$.

39. A nose cone for a space reentry vehicle is designed so that a cross section, taken x ft from the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4}x^2$ ft. Find the volume of the nose cone given that its length is 20 ft.

40. A certain solid is 1 ft high, and a horizontal cross section taken x ft above the bottom of the solid is an annulus of inner radius x^2 ft and outer radius $\sqrt{x}$ ft. Find the volume of the solid.

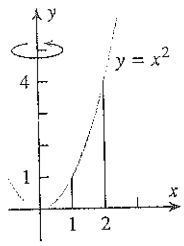
41. Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the x-axis are squares.

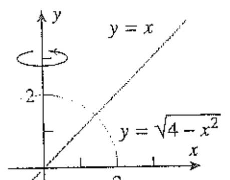
42. The base of a certain solid is the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Every cross section perpendicular to the x-axis is a semicircle with its diameter across the base. Find the volume of the solid.

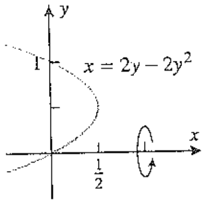
43. In parts (a)–(c) find the volume of the solid whose base is enclosed by the circle $x^2 + y^2 = 1$ and whose cross sections taken perpendicular to the x-axis are as indicated. *(cont.)*

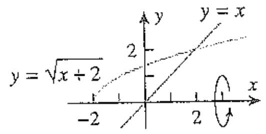
EXERCISE SET 5.3 $\boxed{C}$ CAS

1–4 Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis.

1. 

2. 

3. 

4. 

5–10 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis.

5. $y = x^3,\ x = 1,\ y = 0$
6. $y = \sqrt{x},\ x = 4,\ x = 9,\ y = 0$
7. $y = 1/x,\ y = 0,\ x = 1,\ x = 3$
8. $y = \cos(x^2),\ x = 0,\ x = \frac{1}{2}\sqrt{\pi},\ y = 0$
9. $y = 2x - 1,\ y = -2x + 3,\ x = 2$
10. $y = 2x - x^2,\ y = 0$

11–14 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the x-axis.

11. $y^2 = x,\ y = 1,\ x = 0$
12. $x = 2y,\ y = 2,\ y = 3,\ x = 0$
13. $y = x^2,\ x = 1,\ y = 0$ 14. $xy = 4,\ x + y = 5$

15–18 True–False Determine whether the statement is true or false. Explain your answer.

15. The volume of a cylindrical shell is equal to the product of the thickness of the shell with the surface area of a cylinder whose height is that of the shell and whose radius is equal to the average of the inner and outer radii of the shell.
16. The method of cylindrical shells is a special case of the method of integration of cross-sectional area that was discussed in Section 5.2.
17. In the method of cylindrical shells, integration is over an interval on a coordinate axis that is *perpendicular* to the axis of revolution of the solid.
18. The Riemann sum approximation
$$V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*)\Delta x_k \quad \left(\text{where } x_k^* = \frac{x_k + x_{k-1}}{2}\right)$$
for the volume of a solid of revolution is exact when f is a constant function.

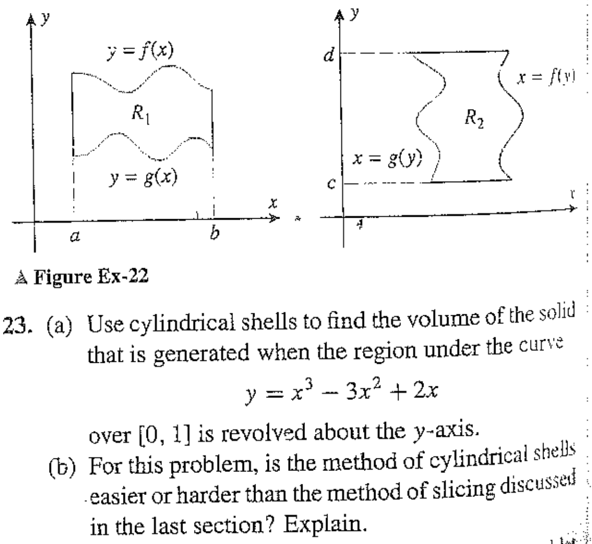
$\boxed{C}$ 19. Use a CAS to find the volume of the solid generated when the region enclosed by $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ is revolved about the y-axis.

$\boxed{C}$ 20. Use a CAS to find the volume of the solid generated when the region enclosed by $y = \cos x$, $y = 0$, and $x = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

$\boxed{C}$ 21. Consider the region to the right of the y-axis, to the left of the vertical line $x = k$ $(0 < k < \pi)$, and between the curve $y = \sin x$ and the x-axis. Use a CAS to estimate the value of k so that the solid generated by revolving the region about the y-axis has a volume of 8 cubic units.

FOCUS ON CONCEPTS

22. Let R_1 and R_2 be regions of the form shown in the accompanying figure. Use cylindrical shells to find a formula for the volume of the solid that results when
(a) region R_1 is revolved about the y-axis
(b) region R_2 is revolved about the x-axis.



▲ **Figure Ex-22**

23. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve
$$y = x^3 - 3x^2 + 2x$$
over $[0, 1]$ is revolved about the y-axis.
(b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.

24. Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is enclosed by $y = f(x)$ and $y = 0$ for $a \le x \le b$. Using the method of cylindrical shells, derive with explanation a formula for the volume of the solid generated by revolving R about the line $x = k$, where $k \le a$.

25–26 Using the method of cylindrical shells, set up but do not evaluate an integral for the volume of the solid generated when the region R is revolved about (a) the line $x = 1$ and (b) the line $y = -1$.

25. R is the region bounded by the graphs of $y = x$, $y = 0$, and $x = 1$.

26. R is the region in the first quadrant bounded by the graphs of $y = \sqrt{1-x^2}$, $y = 0$, and $x = 0$.

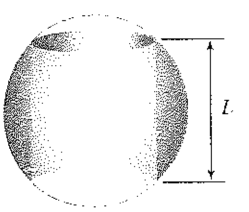
27. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = 1/x^3$, $x = 1$, $x = 2$, $y = 0$ is revolved about the line $x = -1$.

28. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

29. Use cylindrical shells to find the volume of the cone generated when the triangle with vertices $(0, 0)$, $(0, r)$, $(h, 0)$, where $r > 0$ and $h > 0$, is revolved about the x-axis.

30. The region enclosed between the curve $y^2 = kx$ and the line $x = \frac{1}{4}k$ is revolved about the line $x = \frac{1}{2}k$. Use cylindrical shells to find the volume of the resulting solid. (Assume $k > 0$.)

31. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the center of a sphere. Show that the volume of the remaining solid depends only on the length L of the hole, not on the size of the sphere.



◀ Figure Ex-31

32. Use cylindrical shells to find the volume of the torus obtained by revolving the circle $x^2 + y^2 = a^2$ about the line $x = b$, where $b > a > 0$. [*Hint:* It may help in the integration to think of an integral as an area.]

33. Let V_x and V_y be the volumes of the solids that result when the region enclosed by $y = 1/x$, $y = 0$, $x = \frac{1}{2}$, and $x = b$ $(b > \frac{1}{2})$ is revolved about the x-axis and y-axis, respectively. Is there a value of b for which $V_x = V_y$?

34. **Writing** Faced with the problem of computing the volume of a solid of revolution, how would you go about deciding whether to use the method of disks/washers or the method of cylindrical shells?

35. **Writing** With both the method of disks/washers and with the method of cylindrical shells, we integrate an "area" to get the volume of a solid of revolution. However, these two approaches differ in very significant ways. Write a brief paragraph that discusses these differences.

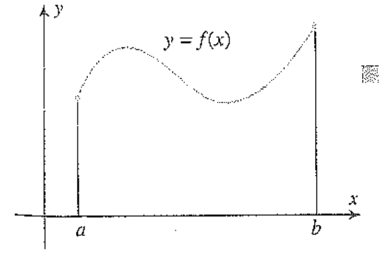
✔ QUICK CHECK ANSWERS 5.3

1. (a) $2\pi x(1+\sqrt{x})$ (b) $\int_1^4 2\pi x(1+\sqrt{x})\,dx$ 2. (a) $2\pi(5-x)(1+\sqrt{x})$ (b) $\int_1^4 2\pi(5-x)(1+\sqrt{x})\,dx$

3. $\int_0^4 2\pi y[4-(y-2)^2]\,dy$

5.4 LENGTH OF A PLANE CURVE

In this section we will use the tools of calculus to study the problem of finding the length of a plane curve.



▲ Figure 5.4.1

ARC LENGTH

Our first objective is to define what we mean by the ***length*** (also called the ***arc length***) of a plane curve $y = f(x)$ over an interval $[a, b]$ (Figure 5.4.1). Once that is done we will be able to focus on the problem of computing arc lengths. To avoid some complications that would otherwise occur, we will impose the requirement that f' be continuous on $[a, b]$, in which case we will say that $y = f(x)$ is a ***smooth curve*** on $[a, b]$ or that f is a ***smooth function*** on $[a, b]$. Thus, we will be concerned with the following problem.