Assignment 6

p 225 # 14, 16, 32

p 286/87 # 12, 20, 24, 32, 44

p 307 # 18

p 321 # 48

p 331 # 24

p 343 # 16

9–10 Evaluate the integral by rewriting the integrand appropriately, if required, and applying the power rule (Formula 2 in Table 4.2.1).

9. (a) $\int x^8\,dx$ (b) $\int x^{5/7}\,dx$ (c) $\int x^3\sqrt{x}\,dx$

10. (a) $\int \sqrt[3]{x^2}\,dx$ (b) $\int \frac{1}{x^6}\,dx$ (c) $\int x^{-7/8}\,dx$

11–14 Evaluate each integral by applying Theorem 4.2.3 and Formula 2 in Table 4.2.1 appropriately.

11. $\int \left[5x + \frac{2}{3x^5}\right] dx$

12. $\int \left[x^{-1/2} - 3x^{7/5} + \frac{1}{9}\right] dx$

13. $\int [x^{-3} - 3x^{1/4} + 8x^2]\,dx$

14. $\int \left[\frac{10}{y^{3/4}} - \sqrt[3]{y} + \frac{4}{\sqrt{y}}\right] dy$

15–30 Evaluate the integral and check your answer by differentiating.

15. $\int x(1+x^3)\,dx$

16. $\int (2+y^2)^2\,dy$

17. $\int x^{1/3}(2-x)^2\,dx$

18. $\int (1+x^2)(2-x)\,dx$

19. $\int \frac{x^5+2x^2-1}{x^4}\,dx$

20. $\int \frac{1-2t^3}{t^3}\,dt$

21. $\int [3\sin x - 2\sec^2 x]\,dx$

22. $\int [\csc^2 t - \sec t \tan t]\,dt$

23. $\int \sec x(\sec x + \tan x)\,dx$

24. $\int \csc x(\sin x + \cot x)\,dx$

25. $\int \frac{\sec\theta}{\cos\theta}\,d\theta$

26. $\int \frac{dy}{\csc y}$

27. $\int \frac{\sin x}{\cos^2 x}\,dx$

28. $\int \left[\phi + \frac{2}{\sin^2\phi}\right] d\phi$

29. $\int [1 + \sin^2\theta \csc\theta]\,d\theta$

30. $\int \frac{\sec x + \cos x}{2\cos x}\,dx$

31. Evaluate the integral

$$\int \frac{1}{1+\sin x}\,dx$$

by multiplying the numerator and denominator by an appropriate expression.

32. Use the double-angle formula $\cos 2x = 2\cos^2 x - 1$ to evaluate the integral

$$\int \frac{1}{1+\cos 2x}\,dx$$

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C$$

34. If C denotes a constant of integration, the two formulas

$$\int \cos x\,dx = \sin x + C$$

$$\int \cos x\,dx = (\sin x + \pi) + C$$

are both correct equations.

35. The function $f(x) = \sec x + 1$ is a solution to the initial-value problem

$$\frac{dy}{dx} = \sec x \tan x, \quad y(0) = 1$$

36. Every integral curve of the slope field

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

is the graph of an increasing function of x.

37. Use a graphing utility to generate some representative integral curves of the function $f(x) = 5x^4 - \sec^2 x$ over the interval $(-\pi/2, \pi/2)$.

38. Use a graphing utility to generate some representative integral curves of the function $f(x) = (x^2-1)/x^2$ over the interval $(0, 5)$.

39–40 Solve the initial-value problems.

39. (a) $\frac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$

(b) $\frac{dy}{dt} = \sin t + 1,\ y\left(\frac{\pi}{3}\right) = \frac{1}{2}$

(c) $\frac{dy}{dx} = \frac{x+1}{\sqrt{x}},\ y(1) = 0$

40. (a) $\frac{dy}{dx} = \frac{1}{(2x)^3},\ y(1) = 0$

(b) $\frac{dy}{dt} = \sec^2 t - \sin t,\ y\left(\frac{\pi}{4}\right) = 1$

(c) $\frac{dy}{dx} = x^2\sqrt{x^3},\ y(0) = 0$

41–44 A particle moves along an s-axis with position function $s = s(t)$ and velocity function $v(t) = s'(t)$. Use the given information to find $s(t)$.

41. $v(t) = 32t;\ s(0) = 20$

42. $v(t) = \cos t;\ s(0) = 2$

43. $v(t) = 3\sqrt{t};\ s(4) = 1$

44. $v(t) = \sin t;\ s(0) = 0$

45. Find the general form of a function whose second derivative is $\sqrt{x}$. [*Hint:* Solve the equation $f''(x) = \sqrt{x}$ for $f(x)$ by integrating both sides twice.]

46. Find a function f such that $f''(x) = x + \cos x$ and such that $f(0) = 1$ and $f'(0) = 2$. [*Hint:* Integrate both sides of the equation twice.]

47–51 Find an equation of the curve that satisfies the given conditions.

47. At each point (x, y) on the curve the slope is $2x + 1$; the curve passes through the point $(-3, 0)$.

48. At each point (x, y) on the curve the slope is $(x+1)^2$; the curve passes through the point $(-2, 8)$.

2. Supply the missing integrand corresponding to the indicated u-substitution.

(a) $\int 5(5x-3)^{-1/3}\,dx = \int \underline{\qquad}\,du;\ u = 5x-3$

(b) $\int (3-\tan x)\sec^2 x\,dx = \int \underline{\qquad}\,du;$ $u = 3-\tan x$

(c) $\int \frac{\sqrt[3]{8+\sqrt{x}}}{\sqrt{x}}\,dx = \int \underline{\qquad}\,du;\ u = 8+\sqrt{x}$

EXERCISE SET 4.3 Graphing Utility [C] CAS

1–8 Evaluate the integrals using the indicated substitutions.

1. (a) $\int 2x(x^2+1)^{23}\,dx;\ u = x^2+1$
   (b) $\int \cos^3 x \sin x\,dx;\ u = \cos x$
2. (a) $\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx;\ u = \sqrt{x}$
   (b) $\int \frac{3x\,dx}{\sqrt{4x^2+5}};\ u = 4x^2+5$
3. (a) $\int \sec^2(4x+1)\,dx;\ u = 4x+1$
   (b) $\int y\sqrt{1+2y^2}\,dy;\ u = 1+2y^2$
4. (a) $\int \sqrt{\sin\pi\theta}\cos\pi\theta\,d\theta;\ u = \sin\pi\theta$
   (b) $\int (2x+7)(x^2+7x+3)^{4/5}\,dx;\ u = x^2+7x+3$
5. (a) $\int \cot x\csc^2 x\,dx;\ u = \cot x$
   (b) $\int (1+\sin t)^9\cos t\,dt;\ u = 1+\sin t$
6. (a) $\int \cos 2x\,dx;\ u = 2x$ (b) $\int x\sec^2 x^2\,dx;\ u = x^2$
7. (a) $\int x^2\sqrt{1+x}\,dx;\ u = 1+x$
   (b) $\int [\csc(\sin x)]^2\cos x\,dx;\ u = \sin x$
8. (a) $\int \sin(x-\pi)\,dx;\ u = x-\pi$
   (b) $\int \frac{5x^4}{(x^5+1)^2}\,dx;\ u = x^5+1$

FOCUS ON CONCEPTS

9. Explain the connection between the chain rule for differentiation and the method of u-substitution for integration.
10. Explain how the substitution $u = ax+b$ helps to perform an integration in which the integrand is $f(ax+b)$, where $f(x)$ is an easy to integrate function.

11–36 Evaluate the integrals using appropriate substitutions.

11. $\int (4x-3)^9\,dx$
12. $\int x^3\sqrt{5+x^4}\,dx$
13. $\int \sin 7x\,dx$
14. $\int \cos\frac{x}{3}\,dx$
15. $\int \sec 4x\tan 4x\,dx$
16. $\int \sec^2 5x\,dx$
17. $\int t\sqrt{7t^2+12}\,dt$
18. $\int \frac{x}{\sqrt{4-5x^2}}\,dx$
19. $\int \frac{6}{(1-2x)^3}\,dx$
20. $\int \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$
21. $\int \frac{x^3}{(5x^4+2)^3}\,dx$
22. $\int \frac{\sin(1/x)}{3x^2}\,dx$
23. $\int \frac{\sin(5/x)}{x^2}\,dx$
24. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$
25. $\int \cos^4 3t\sin 3t\,dt$
26. $\int \cos 2t\sin^5 2t\,dt$
27. $\int x\sec^2(x^2)\,dx$
28. $\int \frac{\cos 4\theta}{(1+2\sin 4\theta)^4}\,d\theta$
29. $\int \cos 4\theta\sqrt{2-\sin 4\theta}\,d\theta$
30. $\int \tan^3 5x\sec^2 5x\,dx$
31. $\int \sec^3 2x\tan 2x\,dx$
32. $\int [\sin(\sin\theta)]\cos\theta\,d\theta$
33. $\int \frac{y}{\sqrt{2y+1}}\,dy$
34. $\int x\sqrt{4-x}\,dx$
35. $\int \sin^3 2\theta\,d\theta$
36. $\int \sec^4 3\theta\,d\theta$ [*Hint:* Apply a trigonometric identity.]

37–39 Evaluate the integrals assuming that n is a positive integer and $b \neq 0$.

37. $\int (a+bx)^n\,dx$
38. $\int \sqrt[n]{a+bx}\,dx$
39. $\int \sin^n(a+bx)\cos(a+bx)\,dx$

[C] 40. Use a CAS to check the answers you obtained in Exercises 37–39. If the answer produced by the CAS does not match yours, show that the two answers are equivalent. [*Suggestion: Mathematica* users may find it helpful to apply the Simplify command to the answer.]

FOCUS ON CONCEPTS

41. (a) Evaluate the integral $\int \sin x \cos x\, dx$ by two methods: first by letting $u = \sin x$, and then by letting $u = \cos x$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

42. (a) Evaluate the integral $\int (5x-1)^2\, dx$ by two methods: first square and integrate, then let $u = 5x - 1$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

43–44 Solve the initial-value problems.

43. $\dfrac{dy}{dx} = \sqrt{5x+1},\ y(3) = -2$

44. $\dfrac{dy}{dx} = 2 + \sin 3x,\ y(\pi/3) = 0$

45. (a) Evaluate $\int [x/\sqrt{x^2+1}]\, dx$.
(b) Use a graphing utility to generate some typical integral curves of $f(x) = x/\sqrt{x^2+1}$ over the interval $(-5, 5)$.

46. (a) Evaluate $\int 2x \sin(25 - x^2)\, dx$.
(b) Use a graphing utility to generate some typical integral curves of $f(x) = 2x \sin(25 - x^2)$ over the interval $(-5, 5)$.

47. Find a function f such that the slope of the tangent line at a point (x, y) on the curve $y = f(x)$ is $\sqrt{3x+1}$ and the curve passes through the point $(0, 1)$.

48. A population of minnows in a lake is estimated to be 100,000 at the beginning of the year 2010. Suppose that t years after the beginning of 2010 the rate of growth of the population $p(t)$ (in thousands) is given by $p'(t) = (3 + 0.12t)^{3/2}$. Estimate the projected population at the beginning of the year 2015.

49. Let $y(t)$ denote the number of *E. coli* cells in a container of nutrient solution t minutes after the start of an experiment. Assume that $y(t)$ is modeled by the initial-value problem

$$\frac{dy}{dt} = 0.95(0.79 + 0.024t)^{3/2}, \qquad y(0) = 20$$

Use this model to estimate the number of *E. coli* cells in the container 20 minutes after the start of the experiment.

50. **Writing** If you want to evaluate an integral by u-substitution, how do you decide what part of the integrand to choose for u?

51. **Writing** The evaluation of an integral can sometimes result in apparently different answers (Exercises 41 and 42). Explain why this occurs and give an example. How might you show that two apparently different answers are actually equivalent?

✓ QUICK CHECK ANSWERS 4.3

1. (a) $1 + x^3$; $3x^2\, dx$ (b) x^2; $2x\, dx$ (c) $1 + 9x^2$; $18x\, dx$ 2. (a) $u^{-1/3}$ (b) $-u$ (c) $2\sqrt[3]{u}$

4.4 THE DEFINITION OF AREA AS A LIMIT; SIGMA NOTATION

Our main goal in this section is to use the rectangle method to give a precise mathematical definition of the "area under a curve."

SIGMA NOTATION

To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called ***sigma notation*** or ***summation notation*** because it uses the uppercase Greek letter Σ (sigma) to denote various kinds of sums. To illustrate how this notation works, consider the sum

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

in which each term is of the form k^2, where k is one of the integers from 1 to 5. In sigma notation this sum can be written as

$$\sum_{k=1}^{5} k^2$$

which is read "the summation of k^2, where k runs from 1 to 5." The notation tells us to form the sum of the terms that result when we substitute successive integers for k in the expression k^2, starting with $k = 1$ and ending with $k = 5$.

EXERCISE SET 4.5

1–4 Find the value of

(a) $\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ (b) $\max \Delta x_k$.

1. $f(x) = x + 1;\ a = 0, b = 4;\ n = 3;$
   $\Delta x_1 = 1, \Delta x_2 = 1, \Delta x_3 = 2;$
   $x_1^* = \frac{1}{3}, x_2^* = \frac{3}{2}, x_3^* = 3$
2. $f(x) = \cos x;\ a = 0, b = 2\pi;\ n = 4;$
   $\Delta x_1 = \pi/2, \Delta x_2 = 3\pi/4, \Delta x_3 = \pi/2, \Delta x_4 = \pi/4;$
   $x_1^* = \pi/4, x_2^* = \pi, x_3^* = 3\pi/2, x_4^* = 7\pi/4$
3. $f(x) = 4 - x^2;\ a = -3, b = 4;\ n = 4;$
   $\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 1, \Delta x_4 = 3;$
   $x_1^* = -\frac{5}{2}, x_2^* = -1, x_3^* = \frac{1}{4}, x_4^* = 3$
4. $f(x) = x^3;\ a = -3, b = 3;\ n = 4;$
   $\Delta x_1 = 2, \Delta x_2 = 1, \Delta x_3 = 1, \Delta x_4 = 2;$
   $x_1^* = -2, x_2^* = 0, x_3^* = 0, x_4^* = 2$

5–8 Use the given values of a and b to express the following limits as integrals. (Do not evaluate the integrals.)

5. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^2 \Delta x_k;\ a = -1, b = 2$
6. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^3 \Delta x_k;\ a = 1, b = 2$
7. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 4x_k^*(1 - 3x_k^*) \Delta x_k;\ a = -3, b = 3$
8. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (\sin^2 x_k^*) \Delta x_k;\ a = 0, b = \pi/2$

9–10 Use Definition 4.5.1 to express the integrals as limits of Riemann sums. (Do not evaluate the integrals.)

9. (a) $\int_1^2 2x\,dx$ (b) $\int_0^1 \frac{x}{x+1}\,dx$
10. (a) $\int_1^2 \sqrt{x}\,dx$ (b) $\int_{-\pi/2}^{\pi/2} (1 + \cos x)\,dx$

FOCUS ON CONCEPTS

11. Explain informally why Theorem 4.5.4(a) follows from Definition 4.5.1.
12. Explain informally why Theorem 4.5.6(a) follows from Definition 4.5.1.

13–16 Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed.

13. (a) $\int_0^3 x\,dx$ (b) $\int_{-2}^{-1} x\,dx$
    (c) $\int_{-1}^{4} x\,dx$ (d) $\int_{-5}^{5} x\,dx$
14. (a) $\int_0^2 \left(1 - \frac{1}{2}x\right)dx$ (b) $\int_{-1}^{1} \left(1 - \frac{1}{2}x\right)dx$
    (c) $\int_2^3 \left(1 - \frac{1}{2}x\right)dx$ (d) $\int_0^3 \left(1 - \frac{1}{2}x\right)dx$
15. (a) $\int_0^5 2\,dx$ (b) $\int_0^{\pi} \cos x\,dx$
    (c) $\int_{-1}^{2} |2x - 3|\,dx$ (d) $\int_{-1}^{1} \sqrt{1 - x^2}\,dx$
16. (a) $\int_{-10}^{-5} 6\,dx$ (b) $\int_{-\pi/3}^{\pi/3} \sin x\,dx$
    (c) $\int_0^3 |x - 2|\,dx$ (d) $\int_0^2 \sqrt{4 - x^2}\,dx$
17. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} |x - 2|, & x \ge 0 \\ x + 2, & x < 0 \end{cases}$$

    (a) $\int_{-2}^{0} f(x)\,dx$ (b) $\int_{-2}^{2} f(x)\,dx$
    (c) $\int_0^6 f(x)\,dx$ (d) $\int_{-4}^{6} f(x)\,dx$
18. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} 2x, & x \le 1 \\ 2, & x > 1 \end{cases}$$

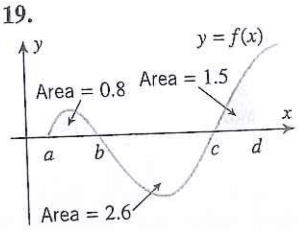
    (a) $\int_0^1 f(x)\,dx$ (b) $\int_{-1}^{1} f(x)\,dx$
    (c) $\int_1^{10} f(x)\,dx$ (d) $\int_{1/2}^{5} f(x)\,dx$

FOCUS ON CONCEPTS

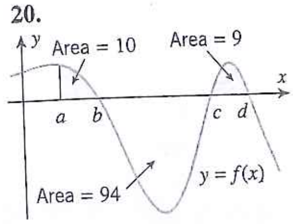
19–20 Use the areas shown in the figure to find

(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$
(c) $\int_a^c f(x)\,dx$ (d) $\int_a^d f(x)\,dx$.

19.



20.



31–34 Use a calculating utility to find the midpoint approximation of the integral using $n = 20$ subintervals, and then find the exact value of the integral using Part 1 of the Fundamental Theorem of Calculus.

31. $\int_1^3 \frac{1}{x^2}\,dx$ **32.** $\int_0^{\pi/2} \sin x\,dx$

33. $\int_{-1}^1 \sec^2 x\,dx$ **34.** $\int_1^5 \frac{1}{x^3}\,dx$

35–38 Sketch the region described and find its area.

35. The region under the curve $y = x^2 + 1$ and over the interval $[0, 3]$.

36. The region below the curve $y = x - x^2$ and above the x-axis.

37. The region under the curve $y = 3\sin x$ and over the interval $[0, 2\pi/3]$.

38. The region below the interval $[-2, -1]$ and above the curve $y = x^3$.

39–42 Sketch the curve and find the total area between the curve and the given interval on the x-axis.

39. $y = x^2 - x$; $[0, 2]$ **40.** $y = \sin x$; $[0, 3\pi/2]$

41. $y = 2\sqrt{x+1} - 3$; $[0, 3]$ **42.** $y = \frac{x^2-1}{x^2}$; $[\frac{1}{2}, 2]$

43. A student wants to find the area enclosed by the graphs of $y = \cos x$, $y = 0$, $x = 0$, and $x = 0.8$.
(a) Show that the exact area is $\sin 0.8$.
(b) The student uses a calculator to approximate the result in part (a) to three decimal places and obtains an incorrect answer of 0.014. What was the student's error? Find the correct approximation.

FOCUS ON CONCEPTS

44. Explain why the Fundamental Theorem of Calculus may be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.

45. (a) If $h'(t)$ is the rate of change of a child's height measured in inches per year, what does the integral $\int_0^{10} h'(t)\,dt$ represent, and what are its units?
(b) If $r'(t)$ is the rate of change of the radius of a spherical balloon measured in centimeters per second, what does the integral $\int_1^2 r'(t)\,dt$ represent, and what are its units?
(c) If $H(t)$ is the rate of change of the speed of sound with respect to temperature measured in ft/s per °F, what does the integral $\int_{32}^{100} H(t)\,dt$ represent, and what are its units?
(d) If $v(t)$ is the velocity of a particle in rectilinear motion, measured in cm/h, what does the integral $\int_{t_1}^{t_2} v(t)\,dt$ represent, and what are its units?

46. (a) Use a graphing utility to generate the graph of
$$f(x) = \frac{1}{100}(x+2)(x+1)(x-3)(x-5)$$
and use the graph to make a conjecture about the sign of the integral
$$\int_{-2}^5 f(x)\,dx$$
(b) Check your conjecture by evaluating the integral.

47. Define $F(x)$ by
$$F(x) = \int_1^x (3t^2 - 3)\,dt$$
(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.
(b) Check the result in part (a) by first integrating and then differentiating.

48. Define $F(x)$ by
$$F(x) = \int_{\pi/4}^x \cos 2t\,dt$$
(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.
(b) Check the result in part (a) by first integrating and then differentiating.

49–52 Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives.

49. (a) $\frac{d}{dx}\int_1^x \sin(t^2)\,dt$ (b) $\frac{d}{dx}\int_1^x \sqrt{1-\cos t}\,dt$

50. (a) $\frac{d}{dx}\int_0^x \frac{dt}{1+\sqrt{t}}$ (b) $\frac{d}{dx}\int_2^x \frac{dt}{t^2+3t-4}$

51. $\frac{d}{dx}\int_x^0 t\sec t\,dt$ [*Hint:* Use Definition 4.5.3(b).]

52. $\frac{d}{du}\int_0^u |x|\,dx$

53. Let $F(x) = \int_4^x \sqrt{t^2+9}\,dt$. Find
(a) $F(4)$ (b) $F'(4)$ (c) $F''(4)$.

54. Let $F(x) = \int_0^x \frac{\cos t}{t^2+3t+5}\,dt$. Find
(a) $F(0)$ (b) $F'(0)$ (c) $F''(0)$.

55. Let $F(x) = \int_0^x \frac{t-3}{t^2+7}\,dt$ for $-\infty < x < +\infty$.
(a) Find the value of x where F attains its minimum value.
(b) Find intervals over which F is only increasing or only decreasing.
(c) Find open intervals over which F is only concave up or only concave down.

56. Use the plotting and numerical integration commands of a CAS to generate the graph of the function F in Exercise 55 over the interval $-20 \le x \le 20$, and confirm that the graph is consistent with the results obtained in that exercise.

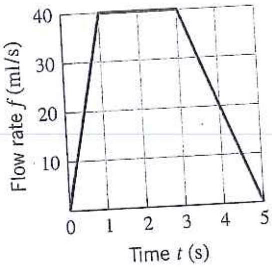
21. Water is run at a constant rate of 1 ft^3/min to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and then check your conjecture by integrating. [Take the weight density of water to be 62.4 lb/ft^3.]

22. (a) The temperature of a 10 m long metal bar is 15°C at one end and 30°C at the other end. Assuming that the temperature increases linearly from the cooler end to the hotter end, what is the average temperature of the bar?
(b) Explain why there must be a point on the bar where the temperature is the same as the average, and find it.

23. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.M. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t) = 100(1 - 0.0001t^2)$ cars per minute, where t is the time (in minutes) since 4:30 P.M. Find the average rate, in cars per minute, at which cars enter the highway during the first half hour of rush hour.

24. Suppose that the value of a yacht in dollars after t years of use is $V(t) = 275{,}000\sqrt{\frac{20}{t+20}}$. What is the average value of the yacht over its first 10 years of use?

25. A large juice glass containing 60 ml of orange juice is replenished by a server. The accompanying figure shows the rate at which orange juice is poured into the glass in milliliters per second (ml/s). Show that the average rate of change of the volume of juice in the glass during these 5 s is equal to the average value of the rate of flow of juice into the glass.



◄ **Figure Ex-25**

C 26. The function J_0 defined by

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t)\, dt$$

is called the ***Bessel function of order zero***.
(a) Find a function f and an interval $[a, b]$ for which $J_0(1)$ is the average value of f over $[a, b]$.
(b) Estimate $J_0(1)$.
(c) Use a CAS to graph the equation $y = J_0(x)$ over the interval $0 \le x \le 8$.
(d) Estimate the smallest positive zero of J_0.

27. Find a positive value of k such that the average value of $f(x) = \sqrt{3x}$ over the interval $[0, k]$ is 6.

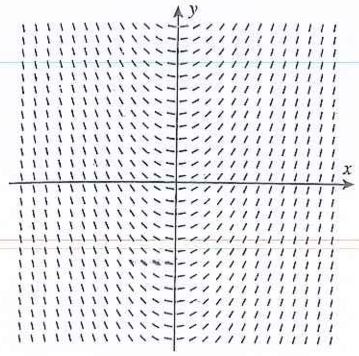
28. Suppose that a tumor grows at the rate of $r(t) = kt$ grams per week for some positive constant k, where t is the number of weeks since the tumor appeared. When, during the second 26 weeks of growth, is the mass of the tumor the same as its average mass during that period?

29. **Writing** Consider the following statement: *The average value of the rate of change of a function over an interval is equal to the average rate of change of the function over that interval.* Write a short paragraph that explains why this statement may be interpreted as a rewording of Part 1 of the Fundamental Theorem of Calculus.

30. **Writing** If an automobile gets an average of 25 miles per gallon of gasoline, then it is also the case that on average the automobile expends 1/25 gallon of gasoline per mile. Interpret this statement using the concept of the average value of a function over an interval.

✔ QUICK CHECK ANSWERS 4.8

1. $\frac{1}{n}\sum_{k=1}^{n} a_k$ 2. $\frac{1}{b-a}\int_a^b f(x)\,dx$ 3. $f(x^*)$ 4. 40



◄ **Figure Ex-6**

7. (a) Show that the substitutions $u = \sec x$ and $u = \tan x$ produce different values for the integral

$$\int \sec^2 x \tan x \, dx$$

(b) Explain why both are correct.

8. Use the two substitutions in Exercise 7 to evaluate the definite integral

$$\int_0^{\pi/4} \sec^2 x \tan x \, dx$$

and confirm that they produce the same result.

9. Evaluate the integral

$$\int \frac{x^7}{\sqrt{x^4+2}} \, dx$$

by making the substitution $u = x^4 + 2$.

10. Evaluate the integral

$$\int \sqrt{1 + x^{-2/3}} \, dx$$

by making the substitution $u = 1 + x^{2/3}$.

C 11–14 Evaluate the integrals by hand, and check your answers with a CAS if you have one. ■

11. $\int \frac{\cos 3x}{\sqrt{5 + 2\sin 3x}} \, dx$ 12. $\int \frac{\sqrt{3+\sqrt{x}}}{\sqrt{x}} \, dx$

13. $\int \frac{x^2}{(ax^3+b)^2} \, dx$ 14. $\int x \sec^2(ax^2) \, dx$

15. Express

$$\sum_{k=4}^{18} k(k-3)$$

in sigma notation with
(a) $k = 0$ as the lower limit of summation
(b) $k = 5$ as the lower limit of summation.

16. (a) Fill in the blank:

$$1 + 3 + 5 + \cdots + (2n-1) = \sum_{k=1}^{n} \_\_\_\_\_\_$$

(b) Use part (a) to prove that the sum of the first n consecutive odd integers is a perfect square.

17. Find the area under the graph of $f(x) = 4x - x^2$ over the interval $[0, 4]$ using Definition 4.4.3 with x_k^* as the *right* endpoint of each subinterval.

18. Find the area under the graph of $f(x) = 5x - x^2$ over the interval $[0, 5]$ using Definition 4.4.3 with x_k^* as the *left* endpoint of each subinterval.

19–20 Use a calculating utility to find the left endpoint, right endpoint, and midpoint approximations to the area under the curve $y = f(x)$ over the stated interval using $n = 10$ subintervals. ■

19. $y = 1/x$; $[1, 2]$ 20. $y = \tan x$; $[0, 1]$

21. The *definite integral* of f over the interval $[a, b]$ is defined as the limit

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Explain what the various symbols on the right side of this equation mean.

22. Use a geometric argument to evaluate

$$\int_0^1 |2x-1| \, dx$$

23. Suppose that

$$\int_0^1 f(x)\,dx = \tfrac{1}{2}, \quad \int_1^2 f(x)\,dx = \tfrac{1}{4},$$

$$\int_0^3 f(x)\,dx = -1, \quad \int_0^1 g(x)\,dx = 2$$

In each part, use this information to evaluate the given integral, if possible. If there is not enough information to evaluate the integral, then say so.

(a) $\int_0^2 f(x)\,dx$ (b) $\int_1^3 f(x)\,dx$ (c) $\int_2^3 5f(x)\,dx$

(d) $\int_1^0 g(x)\,dx$ (e) $\int_0^1 g(2x)\,dx$ (f) $\int_0^1 [g(x)]^2\,dx$

24. In parts (a)–(d), use the information in Exercise 23 to evaluate the given integral. If there is not enough information to evaluate the integral, then say so.

(a) $\int_0^1 [f(x)+g(x)]\,dx$ (b) $\int_0^1 f(x)g(x)\,dx$

(c) $\int_0^1 \frac{f(x)}{g(x)}\,dx$ (d) $\int_0^1 [4g(x) - 3f(x)]\,dx$

25. In each part, evaluate the integral. Where appropriate, you may use a geometric formula.

(a) $\int_{-1}^1 (1 + \sqrt{1-x^2})\,dx$

(b) $\int_0^3 (x\sqrt{x^2+1} - \sqrt{9-x^2})\,dx$

(c) $\int_0^1 x\sqrt{1-x^4}\,dx$