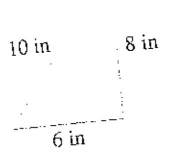
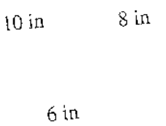
6. A rectangle is to be inscribed in a right triangle having sides of length 6 in, 8 in, and 10 in. Find the dimensions of the rectangle with greatest area assuming the rectangle is positioned as in Figure Ex-6.

7. Solve the problem in Exercise 6 assuming the rectangle is positioned as in Figure Ex-7.



▲ Figure Ex-6



▲ Figure Ex-7

8. A rectangle has its two lower corners on the x-axis and its two upper corners on the curve $y = 16 - x^2$. For all such rectangles, what are the dimensions of the one with largest area?

9. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10.

10. Find the point P in the first quadrant on the curve $y = x^{-2}$ such that a rectangle with sides on the coordinate axes and a vertex at P has the smallest possible perimeter.

11. A rectangular area of 3200 ft^2 is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle of least cost.

12. Show that among all rectangles with perimeter p, the square has the maximum area.

13. Show that among all rectangles with area A, the square has the minimum perimeter.

14. A wire of length 12 in can be bent into a circle, bent into a square, or cut into two pieces to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be
(a) a maximum (b) a minimum?

15. A rectangle R in the plane has corners at $(\pm 8, \pm 12)$, and a 100 by 100 square S is positioned in the plane so that its sides are parallel to the coordinate axes and the lower left corner of S is on the line $y = -3x$. What is the largest possible area of a region in the plane that is contained in both R and S?

16. Solve the problem in Exercise 15 if S is a 16 by 16 square.

17. Solve the problem in Exercise 15 if S is positioned with its lower left corner on the line $y = -6x$.

18. A rectangular page is to contain 42 square inches of printable area. The margins at the top and bottom of the page are each 1 inch, one side margin is 1 inch, and the other side margin is 2 inches. What should the dimensions of the page be so that the least amount of paper is used?

19. A box with a square base is taller than it is wide. In order to send the box through the U.S. mail, the height of the box and the perimeter of the base can sum to no more than 108 in. What is the maximum volume for such a box?

20. A box with a square base is wider than it is tall. In order to send the box through the U.S. mail, the width of the box and the perimeter of one of the (nonsquare) sides of the box can sum to no more than 108 in. What is the maximum volume for such a box?

21. An open box is to be made from a 3 ft by 8 ft rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum volume that the box can have.

22. A closed rectangular container with a square base is to have a volume of 2250 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$3 per in^2. Find the dimensions of the container of least cost.

23. A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.

24. A container with square base, vertical sides, and open top is to be made from 1000 ft^2 of material. Find the dimensions of the container with greatest volume.

25. A rectangular container with two square sides and an open top is to have a volume of V cubic units. Find the dimensions of the container with minimum surface area.

26. A church window consisting of a rectangle topped by a semicircle is to have a perimeter p. Find the radius of the semicircle if the area of the window is to be maximum.

27. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

28. Find the dimensions of the right circular cylinder of greatest surface area that can be inscribed in a sphere of radius R.

29. A closed, cylindrical can is to have a volume of V cubic units. Show that the can of minimum surface area is achieved when the height is equal to the diameter of the base.

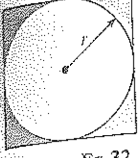
30. A closed cylindrical can is to have a surface area of S square units. Show that the can of maximum volume is achieved when the height is equal to the diameter of the base.

31. A cylindrical can, open at the top, is to hold 500 cm^3 of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

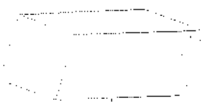
32. A soup can in the shape of a right circular cylinder of radius r and height h is to have a prescribed volume V. The top and bottom are cut from squares as shown in Figure Ex-32 on the next page. If the shaded corners are wasted, but there is no other waste, find the ratio r/h for the can requiring the least material (including waste).

33. A box-shaped wire frame consists of two identical wire squares whose vertices are connected by four straight wires of equal length (Figure Ex-33 on the next page). If

frame is to be made from a wire of length L, what should the dimensions be to obtain a box of greatest volume?



▲ Figure Ex-32

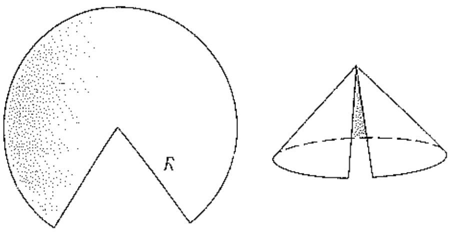


▲ Figure Ex-33

34. Suppose that the sum of the surface areas of a sphere and a cube is a constant.
   (a) Show that the sum of their volumes is smallest when the diameter of the sphere is equal to the length of an edge of the cube.
   (b) When will the sum of their volumes be greatest?

35. Find the height and radius of the cone of slant height L whose volume is as large as possible.

36. A cone is made from a circular sheet of radius R by cutting out a sector and gluing the cut edges of the remaining piece together (Figure Ex-36). What is the maximum volume attainable for the cone?



▲ Figure Ex-36

37. A cone-shaped paper drinking cup is to hold 100 cm^3 of water. Find the height and radius of the cup that will require the least amount of paper.

38. Find the dimensions of the isosceles triangle of least area that can be circumscribed about a circle of radius R.

39. Find the height and radius of the right circular cone with least volume that can be circumscribed about a sphere of radius R.

40. A commercial cattle ranch currently allows 20 steers per acre of grazing land; on the average its steers weigh 2000 lb at market. Estimates by the Agriculture Department indicate that the average market weight per steer will be reduced by 50 lb for each additional steer added per acre of grazing land. How many steers per acre should be allowed in order for the ranch to get the largest possible total market weight for its cattle?

41. A company mines low-grade nickel ore. If the company mines x tons of ore, it can sell the ore for $p = 225 - 0.25x$ dollars per ton. Find the revenue and marginal revenue functions. At what level of production would the company obtain the maximum revenue?

42. A fertilizer producer finds that it can sell its product at a price of $p = 300 - 0.1x$ dollars per unit when it produces x units of fertilizer. The total production cost (in dollars) for x units is
$$C(x) = 15{,}000 + 125x + 0.025x^2$$
If the production capacity of the firm is at most 1000 units of fertilizer in a specified time, how many units must be manufactured and sold in that time to maximize the profit?

43. (a) A chemical manufacturer sells sulfuric acid in bulk at a price of \$100 per unit. If the daily total production cost in dollars for x units is
$$C(x) = 100{,}000 + 50x + 0.0025x^2$$
and if the daily production capacity is at most 7000 units, how many units of sulfuric acid must be manufactured and sold daily to maximize the profit?
   (b) Would it benefit the manufacturer to expand the daily production capacity?
   (c) Use marginal analysis to approximate the effect on profit if daily production could be increased from 7000 to 7001 units.

44. A firm determines that x units of its product can be sold daily at p dollars per unit, where
$$x = 1000 - p$$
The cost of producing x units per day is
$$C(x) = 3000 + 20x$$
   (a) Find the revenue function $R(x)$.
   (b) Find the profit function $P(x)$.
   (c) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.
   (d) Find the maximum profit.
   (e) What price per unit must be charged to obtain the maximum profit?

45. In a certain chemical manufacturing process, the daily weight y of defective chemical output depends on the total weight x of all output according to the empirical formula
$$y = 0.01x + 0.00003x^2$$
where x and y are in pounds. If the profit is \$100 per pound of nondefective chemical produced and the loss is \$20 per pound of defective chemical produced, how many pounds of chemical should be produced daily to maximize the total daily profit?

46. An independent truck driver charges a client \$15 for each hour of driving, plus the cost of fuel. At highway speeds of v miles per hour, the trucker's rig gets $10 - 0.07v$ miles per gallon of diesel fuel. If diesel fuel costs \$2.50 per gallon, what speed v will minimize the cost to the client?

47. A trapezoid is inscribed in a semicircle of radius 2 so that one side is along the diameter (Figure Ex-47 on the next page). Find the maximum possible area for the trapezoid. [*Hint:* Express the area of the trapezoid in terms of θ.]

48. A drainage channel is to be made so that its cross section is a trapezoid with equally sloping sides (Figure Ex-48 on the next page). If the sides and bottom all have a length of 5 ft,

by a distance of 90 cm. Where on the line segment between the two sources is the total intensity a minimum?

62. Given points $A(2, 1)$ and $B(5, 4)$, find the point P in the interval $[2, 5]$ on the x-axis that maximizes angle APB

63. The lower edge of a painting, 10 ft in height, is 2 ft above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?

FOCUS ON CONCEPTS

64. ***Fermat's principle*** (biography on p. 225) in optics states that light traveling from one point to another follows that path for which the total travel time is minimum. In a uniform medium, the paths of "minimum time" and "shortest distance" turn out to be the same, so that light, if unobstructed, travels along a straight line. Assume that we have a light source, a flat mirror, and an observer in a uniform medium. If a light ray leaves the source, bounces off the mirror, and travels on to the observer, then its path will consist of two line segments, as shown in Figure Ex-64. According to Fermat's principle, the path will be such that the total travel time t is minimum or, since the medium is uniform, the path will be such that the total distance traveled from A to P to B is as small as possible. Assuming the minimum occurs when $dt/dx = 0$, show that the light ray will strike the mirror at the point P where the "angle of incidence" θ_1 equals the "angle of reflection" θ_2.

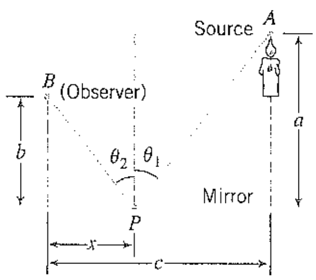


Figure Ex-64

65. Fermat's principle (Exercise 64) also explains why light rays traveling between air and water undergo bending (refraction). Imagine that we have two uniform media (such as air and water) and a light ray traveling from a source A in one medium to an observer B in the other medium (Figure Ex-65). It is known that light travels at a constant speed in a uniform medium, but more slowly in a dense medium (such as water) than in a thin medium (such as air). Consequently, the path of shortest time from A to B is not necessarily a straight line, but rather some broken line path A to P to B allowing the light to take greatest advantage of its higher speed through the thin medium. ***Snell's law of refraction*** (biography on p. 238) states that the path of the light ray will be such that

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

where v_1 is the speed of light in the first medium, v_2 is the speed of light in the second medium, and θ_1 and θ_2 are the angles shown in Figure Ex-65. Show that this follows from the assumption that the path of minimum time occurs when $dt/dx = 0$.

66. A farmer wants to walk at a constant rate from her barn to a straight river, fill her pail, and carry it to her house in the least time.
   (a) Explain how this problem relates to Fermat's principle and the light-reflection problem in Exercise 64.
   (b) Use the result of Exercise 64 to describe geometrically the best path for the farmer to take.
   (c) Use part (b) to determine where the farmer should fill her pail if her house and barn are located as in Figure Ex-66.

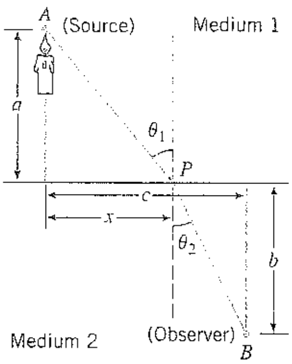


Figure Ex-65

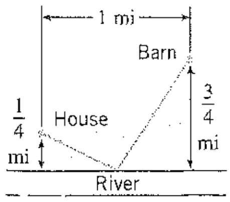


Figure Ex-66

67. If an unknown physical quantity x is measured n times, the measurements $x_1, x_2, \ldots, x_n$ often vary because of uncontrollable factors such as temperature, atmospheric pressure, and so forth. Thus, a scientist is often faced with the problem of using n different observed measurements to obtain an estimate $\bar{x}$ of an unknown quantity x. One method for making such an estimate is based on the ***least squares principle***, which states that the estimate $\bar{x}$ should be chosen to minimize

$$s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2$$

which is the sum of the squares of the deviations between the estimate $\bar{x}$ and the measured values. Show that the estimate resulting from the least squares principle is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

that is, $\bar{x}$ is the arithmetic average of the observed values.

68. Prove: If $f(x) \geq 0$ on an interval and if $f(x)$ has a maximum value on that interval at x_0, then $\sqrt{f(x)}$ also has a maximum value at x_0. Similarly for minimum values. [*Hint:* Use the fact that $\sqrt{x}$ is an increasing function on the interval $[0, +\infty)$.]

69. Writing Discuss the importance of finding intervals of possible values imposed by physical restrictions on variables in an applied maximum or minimum problem.

30. Use the inequality $\sqrt{3} < 1.8$ to prove that

$$1.7 < \sqrt{3} < 1.75$$

[*Hint:* Let $f(x) = \sqrt{x}$, $a = 3$, and $b = 4$ in the Mean-Value Theorem.]

31. Use the Mean-Value Theorem to prove that

$$x - \frac{x^3}{6} < \sin x < x \quad (x > 0)$$

32. Show that if f and g are functions for which

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = f(x)$$

for all x, then $f^2(x) - g^2(x)$ is a constant.

33. (a) Show that if f and g are functions for which

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = -f(x)$$

for all x, then $f^2(x) + g^2(x)$ is a constant.

(b) Give an example of functions f and g with this property.

FOCUS ON CONCEPTS

34. Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Prove: If $f(a) = g(a)$ and $f(b) = g(b)$, then there is a point c in (a, b) such that $f'(c) = g'(c)$.

35. Illustrate the result in Exercise 36 by drawing an appropriate picture.

36. (a) Prove that if $f''(x) > 0$ for all x in (a, b), then $f'(x) = 0$ at most once in (a, b).

(b) Give a geometric interpretation of the result in (a).

37. (a) Prove part (*b*) of Theorem 3.1.2.

(b) Prove part (*c*) of Theorem 3.1.2.

38. Use the Mean-Value Theorem to prove the following result: Let f be continuous at x_0 and suppose that $\lim_{x \to x_0} f'(x)$ exists. Then f is differentiable at x_0, and

$$f'(x_0) = \lim_{x \to x_0} f'(x)$$

[*Hint:* The derivative $f'(x_0)$ is given by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided this limit exists.]

FOCUS ON CONCEPTS

39. Let

$$f(x) = \begin{cases} 3x^2, & x \le 1 \\ ax + b, & x > 1 \end{cases}$$

Find the values of a and b so that f will be differentiable at $x = 1$.

40. (a) Let

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$$

Show that

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$$

but that $f'(0)$ does not exist.

(b) Let

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$$

Show that $f'(0)$ exists but $f''(0)$ does not.

41. Use the Mean-Value Theorem to prove the following result: The graph of a function f has a point of vertical tangency at $(x_0, f(x_0))$ if f is continuous at x_0 and $f'(x)$ approaches either $+\infty$ or $-\infty$ as $x \to x_0^+$ and as $x \to x_0^-$.

42. **Writing** Suppose that $p(x)$ is a nonconstant polynomial with zeros at $x = a$ and $x = b$. Explain how both the Extreme-Value Theorem (3.4.2) and Rolle's Theorem can be used to show that p has a critical point between a and b.

43. **Writing** Find and describe a physical situation that illustrates the Mean-Value Theorem.

✔ QUICK CHECK ANSWERS 3.8

1. (a) $[0, 1]$ (b) $c = \frac{1}{2}$ 2. $[-3, 3]$; $c = -2, 0, 2$ 3. (a) $b = 2$ (b) $c = 1$ 4. (a) 1.5 (b) 0.8 5. $f(x) = x^2 + 4$

CHAPTER 3 REVIEW EXERCISES

Graphing Utility C CAS

1. (a) If $x_1 < x_2$, what relationship must hold between $f(x_1)$ and $f(x_2)$ if f is increasing on an interval containing x_1 and x_2? Decreasing? Constant?

(b) What condition on f' ensures that f is increasing on an interval $[a, b]$? Decreasing? Constant?

2. (a) What condition on f' ensures that f is concave up on an open interval? Concave down?

(b) What condition on f'' ensures that f is concave up on an open interval? Concave down?

(c) In words, what is an inflection point of f?

3–8 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

3. $f(x) = x^2 - 5x + 6$

4. $f(x) = x^4 - 8x^2 + 16$

5. $f(x) = \dfrac{x^2}{x^2 + 2}$

6. $f(x) = \sqrt[3]{x + 2}$

7. $f(x) = x^{1/3}(x + 4)$

8. $f(x) = x^{4/3} - x^{1/3}$

9–12 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility.

9. $f(x) = \cos x$; $[0, 2\pi]$

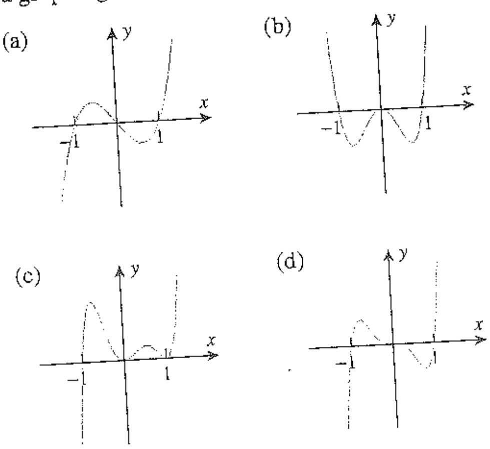
10. $f(x) = \tan x$; $(-\pi/2, \pi/2)$

11. $f(x) = \sin x \cos x$; $[0, \pi]$

12. $f(x) = \cos^2 x - 2 \sin x$; $[0, 2\pi]$

13. In each part, sketch a continuous curve $y = f(x)$ with the stated properties.
   (a) $f(2) = 4$, $f'(2) = 1$, $f''(x) < 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$
   (b) $f(2) = 4$, $f''(x) > 0$ for $x < 2$, $f''(x) < 0$ for $x > 2$, $\lim_{x\to 2^-} f'(x) = +\infty$, $\lim_{x\to 2^+} f'(x) = +\infty$
   (c) $f(2) = 4$, $f''(x) < 0$ for $x \neq 2$, $\lim_{x\to 2^-} f'(x) = 1$, $\lim_{x\to 2^+} f'(x) = -1$

14. In parts (a)–(d), the graph of a polynomial with degree at most 6 is given. Find equations for polynomials that produce graphs with these shapes, and check your answers with a graphing utility.



15. For a general quadratic polynomial
$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$
find conditions on a, b, and c to ensure that f is always increasing or always decreasing on $[0, +\infty)$.

16. For the general cubic polynomial
$$f(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$
find conditions on a, b, c, and d to ensure that f is always increasing or always decreasing on $(-\infty, +\infty)$.

17. (a) Where on the graph of $y = f(x)$ would you expect y to be increasing or decreasing most rapidly with respect to x?
   (b) In words, what is a relative extremum?
   (c) State a procedure for determining where the relative extrema of f occur.

18. Determine whether the statement is true or false. If it is false, give an example for which the statement fails.
   (a) If f has a relative maximum at x_0, then $f(x_0)$ is the largest value that $f(x)$ can have.
   (b) If the largest value for f on the interval (a, b) is at x_0, then f has a relative maximum at x_0.
   (c) A function f has a relative extremum at each of its critical points.

19. (a) According to the first derivative test, what conditions ensure that f has a relative maximum at x_0? A relative minimum?
   (b) According to the second derivative test, what conditions ensure that f has a relative maximum at x_0? A relative minimum?

20–22 Locate the critical points and identify which critical points correspond to stationary points.

20. (a) $f(x) = x^3 + 3x^2 - 9x + 1$
   (b) $f(x) = x^4 - 6x^2 - 3$

21. (a) $f(x) = \dfrac{x}{x^2 + 2}$ (b) $f(x) = \dfrac{x^2 - 3}{x^2 + 1}$

22. (a) $f(x) = x^{1/3}(x - 4)$ (b) $f(x) = x^{4/3} - 6x^{1/3}$

23. In each part, find all critical points, and use the first derivative test to classify them as relative maxima, relative minima, or neither.
   (a) $f(x) = x^{1/3}(x - 7)^2$
   (b) $f(x) = 2 \sin x - \cos 2x$, $0 \le x \le 2\pi$
   (c) $f(x) = 3x - (x - 1)^{3/2}$

24. In each part, find all critical points, and use the second derivative test (where possible) to classify them as relative maxima, relative minima, or neither.
   (a) $f(x) = x^{-1/2} + \frac{1}{9}x^{1/2}$
   (b) $f(x) = x^2 + 8/x$
   (c) $f(x) = \sin^2 x - \cos x$, $0 \le x \le 2\pi$

25–32 Give a graph of the function f, and identify the limits as $x \to \pm\infty$, as well as locations of all relative extrema, inflection points, and asymptotes (as appropriate).

25. $f(x) = x^4 - 3x^3 + 3x^2 + 1$

26. $f(x) = x^5 - 4x^4 + 4x^3$

27. $f(x) = \tan(x^2 + 1)$

28. $f(x) = x - \cos x$

29. $f(x) = \dfrac{x^2}{x^2 + 2x + 5}$

30. $f(x) = \dfrac{25 - 9x^2}{x^3}$

31. $f(x) = \begin{cases} \frac{1}{2}x^2, & x \le 0 \\ -x^2, & x > 0 \end{cases}$

32. $f(x) = (1 + x)^{2/3}(3 - x)^{1/3}$

33–38 Use any method to find the relative extrema of the function f.

33. $f(x) = x^3 + 5x - 2$

34. $f(x) = x^4 - 2x^2 + 7$

35. $f(x) = x^{4/5}$

36. $f(x) = 2x + x^{2/3}$

37. $f(x) = \dfrac{x^2}{x^2+1}$

38. $f(x) = \dfrac{x}{x+2}$

39–40 When using a graphing utility, important features of a graph may be missed if the viewing window is not chosen appropriately. This is illustrated in Exercises 39 and 40.

39. (a) Generate the graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{400}x$ over the interval $[-5, 5]$, and make a conjecture about the locations and nature of all critical points.
   (b) Find the exact locations of all the critical points, and classify them as relative maxima, relative minima, or neither.
   (c) Confirm the results in part (b) by graphing f over an appropriate interval.

40. (a) Generate the graph of
$$f(x) = \tfrac{1}{5}x^5 - \tfrac{7}{8}x^4 + \tfrac{1}{3}x^3 + \tfrac{7}{2}x^2 - 6x$$
over the interval $[-5, 5]$, and make a conjecture about the locations and nature of all critical points.
   (b) Find the exact locations of all the critical points, and classify them as relative maxima, relative minima, or neither.
   (c) Confirm the results in part (b) by graphing portions of f over appropriate intervals. [*Note:* It will not be possible to find a single window in which all of the critical points are discernible.]

41. (a) Use a graphing utility to generate the graphs of $y = x$ and $y = (x^3 - 8)/(x^2 + 1)$ together over the interval $[-5, 5]$, and make a conjecture about the relationship between the two graphs.
   (b) Confirm your conjecture in part (a).

42. Use implicit differentiation to show that a function defined implicitly by $\sin x + \cos y = 2y$ has a critical point whenever $\cos x = 0$. Then use either the first or second derivative test to classify these critical points as relative maxima or minima.

43. Let
$$f(x) = \frac{2x^3 + x^2 - 15x + 7}{(2x-1)(3x^2 + x - 1)}$$
Graph $y = f(x)$, and find the equations of all horizontal and vertical asymptotes. Explain why there is no vertical asymptote at $x = \frac{1}{2}$, even though the denominator of f is zero at that point.

44. Let
$$f(x) = \frac{x^5 - x^4 - 3x^3 + 2x + 4}{x^7 - 2x^6 - 3x^5 + 6x^4 + 4x - 8}$$
   (a) Use a CAS to factor the numerator and denominator of f, and use the results to determine the locations of all vertical asymptotes.
   (b) Confirm that your answer is consistent with the graph of f.

45. (a) What inequality must $f(x)$ satisfy for the function f to have an absolute maximum on an interval I at x_0?
   (b) What inequality must $f(x)$ satisfy for f to have an absolute minimum on an interval I at x_0?
   (c) What is the difference between an absolute extremum and a relative extremum?

46. According to the Extreme-Value Theorem, what conditions on a function f and a given interval guarantee that f will have both an absolute maximum and an absolute minimum on the interval?

47. In each part, determine whether the statement is true or false, and justify your answer.
   (a) If f is differentiable on the open interval (a, b), and if f has an absolute extremum on that interval, then it must occur at a stationary point of f.
   (b) If f is continuous on the open interval (a, b), and if f has an absolute extremum on that interval, then it must occur at a stationary point of f.

48–50 In each part, find the absolute minimum m and the absolute maximum M of f on the given interval (if they exist), and state where the absolute extrema occur.

48. (a) $f(x) = 1/x;\ [-2, -1]$
   (b) $f(x) = x^3 - x^4;\ \left[-1, \frac{3}{2}\right]$
   (c) $f(x) = x - \tan x;\ [-\pi/4, \pi/4]$

49. (a) $f(x) = x^2 - 3x - 1;\ (-\infty, +\infty)$
   (b) $f(x) = x^3 - 3x - 2;\ (-\infty, +\infty)$
   (c) $f(x) = -|x^2 - 2x|;\ [1, 3]$

50. (a) $f(x) = 2x^5 - 5x^4 + 7;\ (-1, 3)$
   (b) $f(x) = (3 - x)/(2 - x);\ (0, 2)$
   (c) $f(x) = 2x/(x^2 + 3);\ (0, 2]$
   (d) $f(x) = x^2(x - 2)^{1/3};\ (0, 3]$

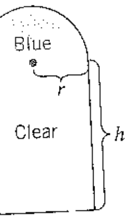
51. In each part, use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values.
   (a) $f(x) = (x^2 - 1)^2;\ (-\infty, +\infty)$
   (b) $f(x) = x/(x^2 + 1);\ [0, +\infty)$
   (c) $f(x) = 2\sec x - \tan x;\ [0, \pi/4]$

52. Prove that $\tan x > x$ for all x in $(0, \pi/2)$.

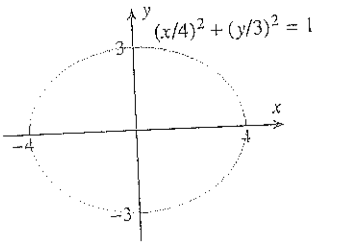
53. Let
$$f(x) = \frac{x^3 + 2}{x^4 + 1}$$
   (a) Generate the graph of $y = f(x)$, and use the graph to make rough estimates of the coordinates of the absolute extrema.
   (b) Use a CAS to solve the equation $f'(x) = 0$ and then use it to make more accurate approximations of the coordinates in part (a).

54. A church window consists of a blue semicircular section surmounting a clear rectangular section as shown in the accompanying figure on the next page. The blue glass lets through half as much light per unit area as the clear glass. Find the radius r of the window that admits the most light if the perimeter of the entire window is to be P feet.

55. Find the dimensions of the rectangle of maximum area that can be inscribed inside the ellipse $(x/4)^2 + (y/3)^2 = 1$ (see the accompanying figure).



▲ Figure Ex-54

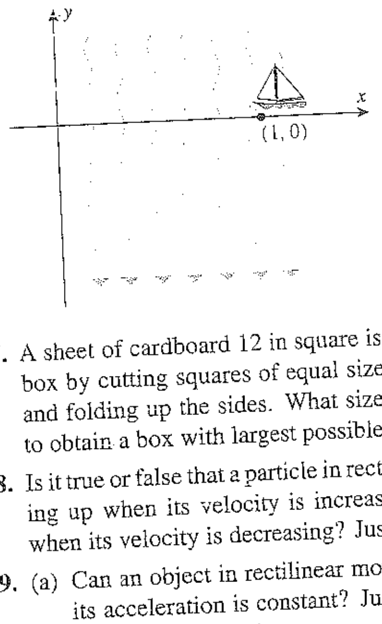


▲ Figure Ex-55

C 56. As shown in the accompanying figure, suppose that a boat enters the river at the point (1, 0) and maintains a heading toward the origin. As a result of the strong current, the boat follows the path

$$y = \frac{x^{10/3} - 1}{2x^{2/3}}$$

where x and y are in miles.
(a) Graph the path taken by the boat.
(b) Can the boat reach the origin? If not, discuss its fate and find how close it comes to the origin.



◄ Figure Ex-56

57. A sheet of cardboard 12 in square is used to make an open box by cutting squares of equal size from the four corners and folding up the sides. What size squares should be cut to obtain a box with largest possible volume?

58. Is it true or false that a particle in rectilinear motion is speeding up when its velocity is increasing and slowing down when its velocity is decreasing? Justify your answer.

59. (a) Can an object in rectilinear motion reverse direction if its acceleration is constant? Justify your answer using a velocity versus time curve.
(b) Can an object in rectilinear motion have increasing speed and decreasing acceleration? Justify your answer using a velocity versus time curve.

60. Suppose that the position function of a particle in rectilinear motion is given by the formula $s(t) = t/(2t^2 + 8)$ for $t \geq 0$.
(a) Use a graphing utility to generate the position, velocity, and acceleration versus time curves.
(b) Use the appropriate graph to make a rough estimate of the time when the particle reverses direction, and then find that time exactly.
(c) Find the position, velocity, and acceleration at the instant when the particle reverses direction.
(d) Use the appropriate graphs to make rough estimates of the time intervals on which the particle is speeding up and the time intervals on which it is slowing down, and then find those time intervals exactly.
(e) When does the particle have its maximum and minimum velocities?

C 61. For parts (a)–(f), suppose that the position function of a particle in rectilinear motion is given by the formula

$$s(t) = \frac{t^2 + 1}{t^4 + 1}, \quad t \geq 0$$

(a) Use a CAS to find simplified formulas for the velocity function $v(t)$ and the acceleration function $a(t)$.
(b) Graph the position, velocity, and acceleration versus time curves.
(c) Use the appropriate graph to make a rough estimate of the time at which the particle is farthest from the origin and its distance from the origin at that time.
(d) Use the appropriate graph to make a rough estimate of the time interval during which the particle is moving in the positive direction.
(e) Use the appropriate graphs to make rough estimates of the time intervals during which the particle is speeding up and the time intervals during which it is slowing down.
(f) Use the appropriate graph to make a rough estimate of the maximum speed of the particle and the time at which the maximum speed occurs.

62. Draw an appropriate picture, and describe the basic idea of Newton's Method without using any formulas.

63. Use Newton's Method to approximate all three solutions of $x^3 - 4x + 1 = 0$.

64. Use Newton's Method to approximate the smallest positive solution of $\sin x + \cos x = 0$.

65. Use a graphing utility to determine the number of times the curve $y = x^3$ intersects the curve $y = (x/2) - 1$. Then apply Newton's Method to approximate the x-coordinates of all intersections.

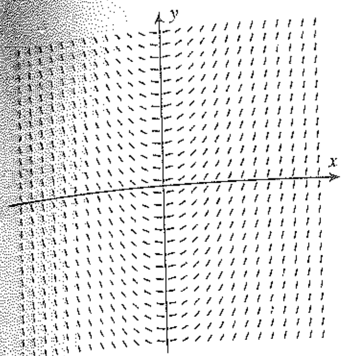
66. According to ***Kepler's law***, the planets in our solar system move in elliptical orbits around the Sun. If a planet's closest approach to the Sun occurs at time $t = 0$, then the distance r from the center of the planet to the center of the Sun at some later time t can be determined from the equation

$$r = a(1 - e\cos\phi)$$

where a is the average distance between centers, e is a positive constant that measures the "flatness" of the elliptical orbit, and ϕ is the solution of *Kepler's equation*

$$\frac{2\pi t}{T} = \phi - e\sin\phi$$

in which T is the time it takes for one complete orbit of the planet. Estimate the distance from the Earth to the Sun



◀ **Figure Ex-6**

7. (a) Show that the substitutions $u = \sec x$ and $u = \tan x$ produce different values for the integral

$$\int \sec^2 x \tan x \, dx$$

(b) Explain why both are correct.

8. Use the two substitutions in Exercise 7 to evaluate the definite integral

$$\int_0^{\pi/4} \sec^2 x \tan x \, dx$$

and confirm that they produce the same result.

9. Evaluate the integral

$$\int \frac{x^7}{\sqrt{x^4+2}} \, dx$$

by making the substitution $u = x^4 + 2$.

10. Evaluate the integral

$$\int \sqrt{1 + x^{-2/3}} \, dx$$

by making the substitution $u = 1 + x^{2/3}$.

C 11–14 Evaluate the integrals by hand, and check your answers with a CAS if you have one.

11. $\displaystyle\int \frac{\cos 3x}{\sqrt{5 + 2\sin 3x}} \, dx$

12. $\displaystyle\int \frac{\sqrt{3+\sqrt{x}}}{\sqrt{x}} \, dx$

13. $\displaystyle\int \frac{x^2}{(ax^3+b)^2} \, dx$

14. $\displaystyle\int x \sec^2(ax^2) \, dx$

15. Express

$$\sum_{k=4}^{18} k(k-3)$$

in sigma notation with
(a) $k = 0$ as the lower limit of summation
(b) $k = 5$ as the lower limit of summation.

16. (a) Fill in the blank:

$$1 + 3 + 5 + \cdots + (2n-1) = \sum_{k=1}^{n} \_\_\_\_\_\_$$

(b) Use part (a) to prove that the sum of the first n consecutive odd integers is a perfect square.

17. Find the area under the graph of $f(x) = 4x - x^2$ over the interval $[0, 4]$ using Definition 4.4.3 with x_k^* as the *right* endpoint of each subinterval.

18. Find the area under the graph of $f(x) = 5x - x^2$ over the interval $[0, 5]$ using Definition 4.4.3 with x_k^* as the *left* endpoint of each subinterval.

19–20 Use a calculating utility to find the left endpoint, right endpoint, and midpoint approximations to the area under the curve $y = f(x)$ over the stated interval using $n = 10$ subintervals.

19. $y = 1/x$; $[1, 2]$

20. $y = \tan x$; $[0, 1]$

21. The *definite integral* of f over the interval $[a, b]$ is defined as the limit

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Explain what the various symbols on the right side of this equation mean.

22. Use a geometric argument to evaluate

$$\int_0^1 |2x - 1| \, dx$$

23. Suppose that

$$\int_0^1 f(x)\,dx = \tfrac{1}{2}, \quad \int_1^2 f(x)\,dx = \tfrac{1}{4},$$
$$\int_0^3 f(x)\,dx = -1, \quad \int_0^1 g(x)\,dx = 2$$

In each part, use this information to evaluate the given integral, if possible. If there is not enough information to evaluate the integral, then say so.

(a) $\displaystyle\int_0^2 f(x)\,dx$ (b) $\displaystyle\int_1^3 f(x)\,dx$ (c) $\displaystyle\int_2^3 5f(x)\,dx$

(d) $\displaystyle\int_1^0 g(x)\,dx$ (e) $\displaystyle\int_0^1 g(2x)\,dx$ (f) $\displaystyle\int_0^1 [g(x)]^2\,dx$

24. In parts (a)–(d), use the information in Exercise 23 to evaluate the given integral. If there is not enough information to evaluate the integral, then say so.

(a) $\displaystyle\int_0^1 [f(x) + g(x)]\,dx$ (b) $\displaystyle\int_0^1 f(x)g(x)\,dx$

(c) $\displaystyle\int_0^1 \frac{f(x)}{g(x)}\,dx$ (d) $\displaystyle\int_0^1 [4g(x) - 3f(x)]\,dx$

25. In each part, evaluate the integral. Where appropriate, you may use a geometric formula.

(a) $\displaystyle\int_{-1}^{1} (1 + \sqrt{1-x^2})\,dx$

(b) $\displaystyle\int_0^3 (x\sqrt{x^2+1} - \sqrt{9-x^2})\,dx$

(c) $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx$

26. In each part, find the limit by interpreting it as a limit of Riemann sums in which the interval [0, 1] is divided into n subintervals of equal length.

(a) $\lim_{n\to+\infty} \dfrac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n^{3/2}}$

(b) $\lim_{n\to+\infty} \dfrac{1^4+2^4+3^4+\cdots+n^4}{n^5}$

27–34 Evaluate the integrals using the Fundamental Theorem of Calculus and (if necessary) properties of the definite integral.

27. $\int_{-3}^{0} (x^2 - 4x + 7)\,dx$

28. $\int_{-1}^{2} x(1+x^3)\,dx$

29. $\int_{1}^{3} \dfrac{1}{x^2}\,dx$

30. $\int_{1}^{8} (5x^{2/3} - 4x^{-2})\,dx$

31. $\int_{0}^{1} (x - \sec x \tan x)\,dx$

32. $\int_{1}^{4} \left(\dfrac{3}{\sqrt{t}} - 5\sqrt{t} - t^{-3/2}\right) dt$

33. $\int_{0}^{2} |2x-3|\,dx$

34. $\int_{0}^{\pi/2} \left|\tfrac{1}{2} - \sin x\right|\,dx$

35–36 Find the area under the curve $y = f(x)$ over the stated interval.

35. $f(x) = \sqrt{x};\ [1, 9]$

36. $f(x) = x^{-3/5};\ [1, 4]$

37. Find the area that is above the x-axis but below the curve $y = (1-x)(x-2)$. Make a sketch of the region.

[C] 38. Use a CAS to find the area of the region in the first quadrant that lies below the curve $y = x + x^2 - x^3$ and above the x-axis.

39–40 Sketch the curve and find the total area between the curve and the given interval on the x-axis.

39. $y = x^2 - 1;\ [0, 3]$

40. $y = \sqrt{x+1} - 1;\ [-1, 1]$

41. Define $F(x)$ by

$$F(x) = \int_{1}^{x} (t^3 + 1)\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

42. Define $F(x)$ by

$$F(x) = \int_{4}^{x} \frac{1}{\sqrt{t}}\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

43–46 Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives.

43. $\dfrac{d}{dx}\left[\int_{0}^{x} \dfrac{1}{t^4+5}\,dt\right]$

44. $\dfrac{d}{dx}\left[\int_{0}^{x} \dfrac{t}{\cos t^2}\,dt\right]$

45. $\dfrac{d}{dx}\left[\int_{0}^{x} |t-1|\,dt\right]$

46. $\dfrac{d}{dx}\left[\int_{\pi}^{x} \cos\sqrt{t}\,dt\right]$

47. State the two parts of the Fundamental Theorem of Calculus, and explain what is meant by the statement "Differentiation and integration are inverse processes."

[C] 48. Let $F(x) = \int_{0}^{x} \dfrac{t^2-3}{t^4+7}\,dt.$

(a) Find the intervals on which F is increasing and those on which F is decreasing.

(b) Find the open intervals on which F is concave up and those on which F is concave down.

(c) Find the x-values, if any, at which the function F has absolute extrema.

(d) Use a CAS to graph F, and confirm that the results in parts (a), (b), and (c) are consistent with the graph.

49. Use differentiation to prove that the function

$$F(x) = \int_{0}^{x} \frac{1}{1+t^2}\,dt + \int_{0}^{1/x} \frac{1}{1+t^2}\,dt$$

is constant on the interval $(0, +\infty)$.

50. What is the natural domain of the function

$$F(x) = \int_{1}^{x} \frac{1}{t^2-9}\,dt?$$

Explain your reasoning.

51. In each part, determine the values of x for which $F(x)$ is positive, negative, or zero without performing the integration; explain your reasoning.

(a) $F(x) = \int_{1}^{x} \dfrac{t^4}{t^2+3}\,dt$ (b) $F(x) = \int_{-1}^{x} \sqrt{4-t^2}\,dt$

[C] 52. Use a CAS to approximate the largest and smallest values of the integral

$$\int_{-1}^{x} \frac{t}{\sqrt{2+t^3}}\,dt$$

for $1 \le x \le 3$.

53. Find all values of x^* in the stated interval that are guaranteed to exist by the Mean-Value Theorem for Integrals, and explain what these numbers represent.

(a) $f(x) = \sqrt{x};\ [0, 3]$ (b) $f(x) = 2x - x^2;\ [0, 2]$

54. A 10-gram tumor is discovered in a laboratory rat on March 1. The tumor is growing at a rate of $r(t) = t/7$ grams per week, where t denotes the number of weeks since March 1. What will be the mass of the tumor on June 7?

55. Use the graph of f shown in the accompanying figure on the next page to find the average value of f on the interval [0, 10].