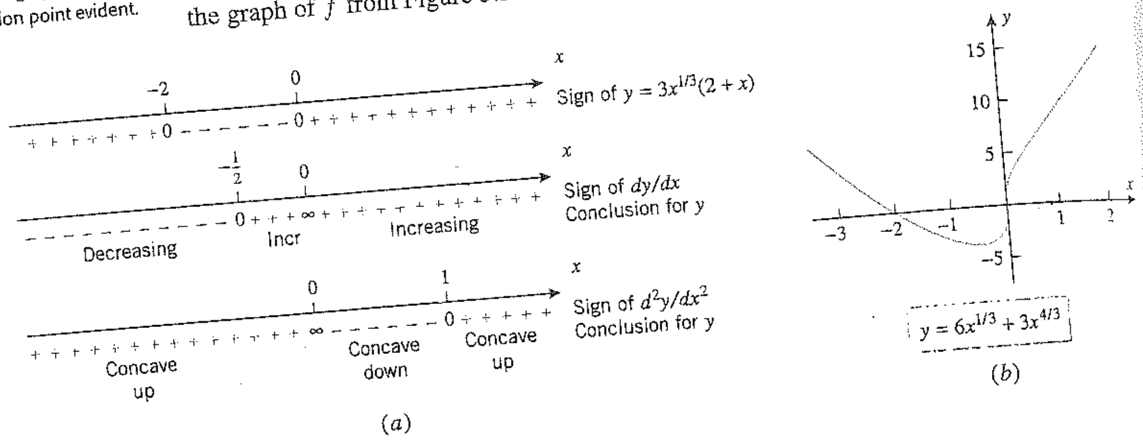
- From the formula for dy/dx we see that there is a stationary point at $x = -\frac{1}{2}$ and a critical point at $x = 0$ at which f is not differentiable. We saw above that a vertical tangent line and inflection point are at that critical point.
- The sign analysis of dy/dx in Figure 3.3.7a and the first derivative test show that there is a relative minimum at the stationary point at $x = -\frac{1}{2}$ (verify).
- The sign analysis of d^2y/dx^2 in Figure 3.3.7a shows that in addition to the inflection point at the vertical tangent there is an inflection point at $x = 1$ at which the graph changes from concave down to concave up.

These conclusions reinforce, and in some cases extend, the information we inferred about the graph of f from Figure 3.3.7b. ◄

TECHNOLOGY MASTERY

The graph in Figure 3.3.7b was generated with a graphing utility. However, the inflection point at $x = 1$ is so subtle that it is not evident from this graph. See if you can produce a version of this graph with your graphing utility that makes the inflection point evident.



▲ Figure 3.3.7

✔ QUICK CHECK EXERCISES 3.3 (See page 216 for answers.)

1. Let $f(x) = \dfrac{3(x+1)(x-3)}{(x+2)(x-4)}$. Given that

$$f'(x) = \frac{-30(x-1)}{(x+2)^2(x-4)^2}, \qquad f''(x) = \frac{90(x^2-2x+4)}{(x+2)^3(x-4)^3}$$

determine the following properties of the graph of f.
(a) The x- and y-intercepts are ______.
(b) The vertical asymptotes are ______.
(c) The horizontal asymptote is ______.
(d) The graph is above the x-axis on the intervals ______.
(e) The graph is increasing on the intervals ______.
(f) The graph is concave up on the intervals ______.
(g) The relative maximum point on the graph is ______.

2. Let $f(x) = \dfrac{x^2-4}{x^{8/3}}$. Given that

$$f'(x) = \frac{-2(x^2-16)}{3x^{11/3}}, \qquad f''(x) = \frac{2(5x^2-176)}{9x^{14/3}}$$

determine the following properties of the graph of f.
(a) The x-intercepts are ______.
(b) The vertical asymptote is ______.
(c) The horizontal asymptote is ______.
(d) The graph is above the x-axis on the intervals ______.
(e) The graph is increasing on the intervals ______.
(f) The graph is concave up on the intervals ______.
(g) Inflection points occur at $x =$ ______.

EXERCISE SET 3.3 Graphing Utility

1–14 Give a graph of the rational function and label the coordinates of the stationary points and inflection points. Show the horizontal and vertical asymptotes and label them with their equations. Label point(s), if any, where the graph crosses a horizontal asymptote. Check your work with a graphing utility.

1. $\dfrac{2x-6}{4-x}$
2. $\dfrac{8}{x^2-4}$
3. $\dfrac{x}{x^2-4}$
4. $\dfrac{x^2}{x^2-4}$
5. $\dfrac{x^2}{x^2+4}$
6. $\dfrac{(x^2-1)^2}{x^4+1}$
7. $\dfrac{x^3+1}{x^3-1}$
8. $2 - \dfrac{1}{3x^2+x^3}$
9. $\dfrac{4}{x^2} - \dfrac{2}{x} + 3$
10. $\dfrac{3(x+1)^2}{(x-1)^2}$
11. $\dfrac{(3x+1)^2}{(x-1)^2}$
12. $3 + \dfrac{x+1}{(x-1)^4}$
13. $\dfrac{x^2+x}{1-x^2}$
14. $\dfrac{x^2}{1-x^3}$

15–16 In each part, make a rough sketch of the graph using asymptotes and appropriate limits but no derivatives. Compare your graph to that generated with a graphing utility.

15. (a) $y = \dfrac{3x^2 - 8}{x^2 - 4}$ (b) $y = \dfrac{x^2 + 2x}{x^2 - 1}$

16. (a) $y = \dfrac{2x - x^2}{x^2 + x - 2}$ (b) $y = \dfrac{x^2}{x^2 - x - 2}$

17. Show that $y = x + 3$ is an oblique asymptote of the graph of $f(x) = x^2/(x - 3)$. Sketch the graph of $y = f(x)$ showing this asymptotic behavior.

18. Show that $y = 3 - x^2$ is a curvilinear asymptote of the graph of $f(x) = (2 + 3x - x^3)/x$. Sketch the graph of $y = f(x)$ showing this asymptotic behavior.

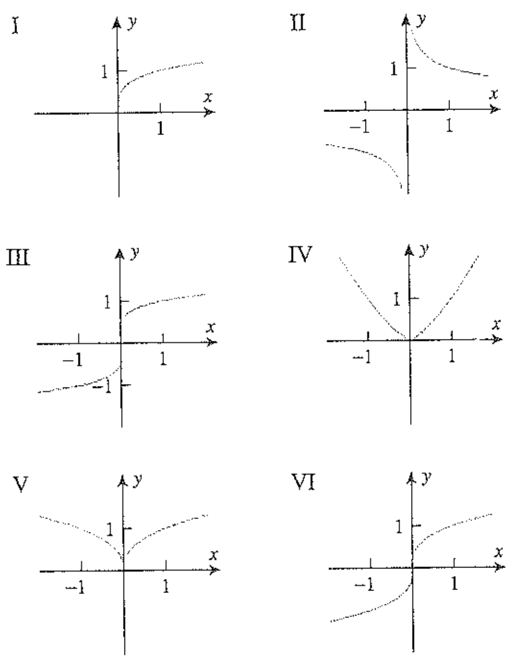
19–24 Sketch a graph of the rational function and label the coordinates of the stationary points and inflection points. Show the horizontal, vertical, oblique, and curvilinear asymptotes and label them with their equations. Label point(s), if any, where the graph crosses an asymptote. Check your work with a graphing utility.

19. $x^2 - \dfrac{1}{x}$ 20. $\dfrac{x^2 - 2}{x}$ 21. $\dfrac{(x - 2)^3}{x^2}$

22. $x - \dfrac{1}{x} - \dfrac{1}{x^2}$ 23. $\dfrac{x^3 - 4x - 8}{x + 2}$ 24. $\dfrac{x^5}{x^2 + 1}$

FOCUS ON CONCEPTS

25. In each part, match the function with graphs I–VI.
(a) $x^{1/3}$ (b) $x^{1/4}$ (c) $x^{1/5}$
(d) $x^{2/5}$ (e) $x^{4/3}$ (f) $x^{-1/3}$



▲ Figure Ex-25

26. Sketch the general shape of the graph of $y = x^{1/n}$, and then explain in words what happens to the shape of the graph as n increases if
(a) n is a positive even integer
(b) n is a positive odd integer.

27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. Suppose that $f(x) = P(x)/Q(x)$, where P and Q are polynomials with no common factors. If $y = 5$ is a horizontal asymptote for the graph of f, then P and Q have the same degree.

28. If the graph of f has a vertical asymptote at $x = 1$, then f cannot be continuous at $x = 1$.

29. If the graph of f' has a vertical asymptote at $x = 1$, then f cannot be continuous at $x = 1$.

30. If the graph of f has a cusp at $x = 1$, then f cannot have an inflection point at $x = 1$.

31–38 Give a graph of the function and identify the locations of all critical points and inflection points. Check your work with a graphing utility.

31. $\sqrt{4x^2 - 1}$ 32. $\sqrt[3]{x^2 - 4}$

33. $2x + 3x^{2/3}$ 34. $2x^2 - 3x^{4/3}$

35. $4x^{1/3} - x^{4/3}$ 36. $5x^{2/3} + x^{5/3}$

37. $\dfrac{8 + x}{2 + \sqrt[3]{x}}$ 38. $\dfrac{8(\sqrt{x} - 1)}{x}$

39–44 Give a graph of the function and identify the locations of all relative extrema and inflection points. Check your work with a graphing utility.

39. $x + \sin x$ 40. $x - \tan x$

41. $\sqrt{3}\cos x + \sin x$ 42. $\sin x + \cos x$

43. $\sin^2 x - \cos x, \quad -\pi \le x \le 3\pi$

44. $\sqrt{\tan x}, \quad 0 \le x < \pi/2$

FOCUS ON CONCEPTS

45. The accompanying figure on the next page shows the graph of the *derivative* of a function h that is defined and continuous on the interval $(-\infty, +\infty)$. Assume that the graph of h' has a vertical asymptote at $x = 3$ and that

$$h'(x) \to 0^+ \text{ as } x \to -\infty$$
$$h'(x) \to -\infty \text{ as } x \to +\infty$$

(a) What are the critical points for $h(x)$?
(b) Identify the intervals on which $h(x)$ is increasing.
(c) Identify the x-coordinates of relative extrema for $h(x)$ and classify each as a relative maximum or relative minimum.
(d) Estimate the x-coordinates of inflection points for $h(x)$.

21–28 Find the absolute maximum and minimum values of f, if any, on the given interval, and state where those values occur.

21. $f(x) = x^2 - x - 2;\ (-\infty, +\infty)$
22. $f(x) = 3 - 4x - 2x^2;\ (-\infty, +\infty)$
23. $f(x) = 4x^3 - 3x^4;\ (-\infty, +\infty)$
24. $f(x) = x^4 + 4x;\ (-\infty, +\infty)$
25. $f(x) = 2x^3 - 6x + 2;\ (-\infty, +\infty)$
26. $f(x) = x^3 - 9x + 1;\ (-\infty, +\infty)$
27. $f(x) = \dfrac{x^2+1}{x+1};\ (-5, -1)$
28. $f(x) = \dfrac{x-2}{x+1};\ (-1, 5]$

29–38 Use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values.

29. $f(x) = (x^2 - 2x)^2;\ (-\infty, +\infty)$
30. $f(x) = (x-1)^2(x+2)^2;\ (-\infty, +\infty)$
31. $f(x) = x^{2/3}(20 - x);\ [-1, 20]$
32. $f(x) = \dfrac{x}{x^2+2};\ [-1, 4]$
33. $f(x) = 1 + \dfrac{1}{x};\ (0, +\infty)$
34. $f(x) = \dfrac{2x^2 - 3x + 3}{x^2 - 2x + 2};\ [1, +\infty)$
35. $f(x) = \dfrac{2 - \cos x}{\sin x};\ [\pi/4, 3\pi/4]$
36. $f(x) = \sin^2 x + \cos x;\ [-\pi, \pi]$
37. $f(x) = \sin(\cos x);\ [0, 2\pi]$
38. $f(x) = \cos(\sin x);\ [0, \pi]$
39. Find the absolute maximum and minimum values of

$$f(x) = \begin{cases} 4x - 2, & x < 1 \\ (x-2)(x-3), & x \geq 1 \end{cases}$$

on $\left[\frac{1}{2}, \frac{7}{2}\right]$.

40. Let $f(x) = x^2 + px + q$. Find the values of p and q such that $f(1) = 3$ is an extreme value of f on $[0, 2]$. Is this value a maximum or minimum?

41–42 If f is a periodic function, then the locations of all absolute extrema on the interval $(-\infty, +\infty)$ can be obtained by finding the locations of the absolute extrema for one period and using the periodicity to locate the rest. Use this idea in these exercises to find the absolute maximum and minimum values of the function, and state the x-values at which they occur.

41. $f(x) = 2\cos x + \cos 2x$
42. $f(x) = 3\cos\dfrac{x}{3} + 2\cos\dfrac{x}{2}$

43–44 One way of proving that $f(x) \leq g(x)$ for all x in a given interval is to show that $0 \leq g(x) - f(x)$ for all x in the interval; and one way of proving the latter inequality is to show that the absolute minimum value of $g(x) - f(x)$ on the interval is nonnegative. Use this idea to prove the inequalities in these exercises.

43. Prove that $\sin x \leq x$ for all x in the interval $[0, 2\pi]$.
44. Prove that $\cos x \geq 1 - (x^2/2)$ for all x in the interval $[0, 2\pi]$.
45. What is the smallest possible slope for a tangent to the graph of the equation $y = x^3 - 3x^2 + 5x$?
46. (a) Show that $f(x) = \sec x + \csc x$ has a minimum value but no maximum value on the interval $(0, \pi/2)$.
(b) Find the minimum value in part (a).

C 47. Show that the absolute minimum value of

$$f(x) = x^2 + \frac{x^2}{(8-x)^2}, \quad x > 8$$

occurs at $x = 10$ by using a CAS to find $f'(x)$ and to solve the equation $f'(x) = 0$.

C 48. The vertical displacement $f(t)$ of a cork bobbing up and down on the ocean's surface may be modeled by the function

$$f(t) = A\cos t + B\sin t$$

where $A > 0$ and $B > 0$. Use a CAS to find the maximum and minimum values of $f(t)$ in terms of A and B.

49. Suppose that the equations of motion of a paper airplane during the first 12 seconds of flight are

$$x = t - 2\sin t, \quad y = 2 - 2\cos t \quad (0 \leq t \leq 12)$$

What are the highest and lowest points in the trajectory, and when is the airplane at those points?

50. The accompanying figure shows the path of a fly whose equations of motion are

$$x = \frac{\cos t}{2 + \sin t}, \quad y = 3 + \sin(2t) - 2\sin^2 t \quad (0 \leq t \leq 2\pi)$$

(a) How high and low does it fly?
(b) How far left and right of the origin does it fly?

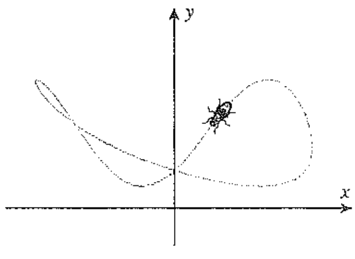
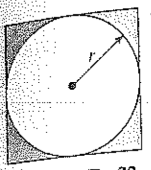


Figure Ex-50

51. Let $f(x) = ax^2 + bx + c$, where $a > 0$. Prove that $f(x) \geq 0$ for all x if and only if $b^2 - 4ac \leq 0$. [*Hint:* Find the minimum of $f(x)$.]
52. Prove Theorem 3.4.3 in the case where the extreme value is a minimum.
53. **Writing** Suppose that f is continuous and positive-valued everywhere and that the x-axis is an asymptote for the graph

frame is to be made from a wire of length L, what should the dimensions be to obtain a box of greatest volume?



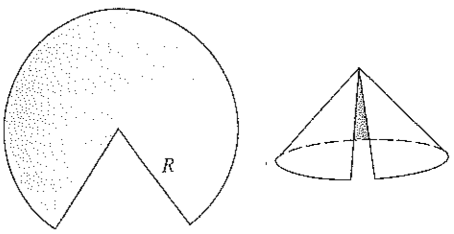
▲ Figure Ex-32

▲ Figure Ex-33

34. Suppose that the sum of the surface areas of a sphere and a cube is a constant.
   (a) Show that the sum of their volumes is smallest when the diameter of the sphere is equal to the length of an edge of the cube.
   (b) When will the sum of their volumes be greatest?

35. Find the height and radius of the cone of slant height L whose volume is as large as possible.

36. A cone is made from a circular sheet of radius R by cutting out a sector and gluing the cut edges of the remaining piece together (Figure Ex-36). What is the maximum volume attainable for the cone?



▲ Figure Ex-36

37. A cone-shaped paper drinking cup is to hold 100 cm^3 of water. Find the height and radius of the cup that will require the least amount of paper.

38. Find the dimensions of the isosceles triangle of least area that can be circumscribed about a circle of radius R.

39. Find the height and radius of the right circular cone with least volume that can be circumscribed about a sphere of radius R.

40. A commercial cattle ranch currently allows 20 steers per acre of grazing land; on the average its steers weigh 2000 lb at market. Estimates by the Agriculture Department indicate that the average market weight per steer will be reduced by 50 lb for each additional steer added per acre of grazing land. How many steers per acre should be allowed in order for the ranch to get the largest possible total market weight for its cattle?

41. A company mines low-grade nickel ore. If the company mines x tons of ore, it can sell the ore for $p = 225 - 0.25x$ dollars per ton. Find the revenue and marginal revenue functions. At what level of production would the company obtain the maximum revenue?

42. A fertilizer producer finds that it can sell its product at a price of $p = 300 - 0.1x$ dollars per unit when it produces x units of fertilizer. The total production cost (in dollars) for x units is

$$C(x) = 15{,}000 + 125x + 0.025x^2$$

If the production capacity of the firm is at most 1000 units of fertilizer in a specified time, how many units must be manufactured and sold in that time to maximize the profit?

43. (a) A chemical manufacturer sells sulfuric acid in bulk at a price of \$100 per unit. If the daily total production cost in dollars for x units is

$$C(x) = 100{,}000 + 50x + 0.0025x^2$$

   and if the daily production capacity is at most 7000 units, how many units of sulfuric acid must be manufactured and sold daily to maximize the profit?
   (b) Would it benefit the manufacturer to expand the daily production capacity?
   (c) Use marginal analysis to approximate the effect on profit if daily production could be increased from 7000 to 7001 units.

44. A firm determines that x units of its product can be sold daily at p dollars per unit, where

$$x = 1000 - p$$

The cost of producing x units per day is

$$C(x) = 3000 + 20x$$

   (a) Find the revenue function $R(x)$.
   (b) Find the profit function $P(x)$.
   (c) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.
   (d) Find the maximum profit.
   (e) What price per unit must be charged to obtain the maximum profit?

45. In a certain chemical manufacturing process, the daily weight y of defective chemical output depends on the total weight x of all output according to the empirical formula

$$y = 0.01x + 0.00003x^2$$

where x and y are in pounds. If the profit is \$100 per pound of nondefective chemical produced and the loss is \$20 per pound of defective chemical produced, how many pounds of chemical should be produced daily to maximize the total daily profit?

46. An independent truck driver charges a client \$15 for each hour of driving, plus the cost of fuel. At highway speeds of v miles per hour, the trucker's rig gets $10 - 0.07v$ miles per gallon of diesel fuel. If diesel fuel costs \$2.50 per gallon, what speed v will minimize the cost to the client?

47. A trapezoid is inscribed in a semicircle of radius 2 so that one side is along the diameter (Figure Ex-47 on the next page). Find the maximum possible area for the trapezoid. [*Hint:* Express the area of the trapezoid in terms of θ.]

48. A drainage channel is to be made so that its cross section is a trapezoid with equally sloping sides (Figure Ex-48 on the next page). If the sides and bottom all have a length of 5 ft,

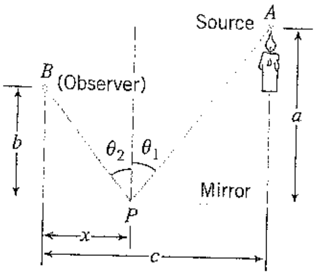
by a distance of 90 cm. Where on the line segment between the two sources is the total intensity a minimum?

62. Given points $A(2, 1)$ and $B(5, 4)$, find the point P in the interval $[2, 5]$ on the x-axis that maximizes angle APB.

63. The lower edge of a painting, 10 ft in height, is 2 ft above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?

FOCUS ON CONCEPTS

64. ***Fermat's principle*** (biography on p. 225) in optics states that light traveling from one point to another follows that path for which the total travel time is minimum. In a uniform medium, the paths of "minimum time" and "shortest distance" turn out to be the same, so that light, if unobstructed, travels along a straight line. Assume that we have a light source, a flat mirror, and an observer in a uniform medium. If a light ray leaves the source, bounces off the mirror, and travels on to the observer, then its path will consist of two line segments, as shown in Figure Ex-64. According to Fermat's principle, the path will be such that the total travel time t is minimum or, since the medium is uniform, the path will be such that the total distance traveled from A to P to B is as small as possible. Assuming the minimum occurs when $dt/dx = 0$, show that the light ray will strike the mirror at the point P where the "angle of incidence" θ_1 equals the "angle of reflection" θ_2.



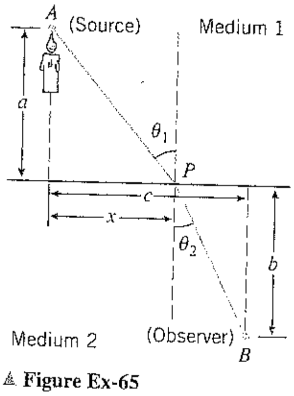
◄ **Figure Ex-64**

65. Fermat's principle (Exercise 64) also explains why light rays traveling between air and water undergo bending (refraction). Imagine that we have two uniform media (such as air and water) and a light ray traveling from a source A in one medium to an observer B in the other medium (Figure Ex-65). It is known that light travels at a constant speed in a uniform medium, but more slowly in a dense medium (such as water) than in a thin medium (such as air). Consequently, the path of shortest time from A to B is not necessarily a straight line, but rather some broken line path A to P to B allowing the light to take greatest advantage of its higher speed through the thin medium. ***Snell's law of refraction*** (biography on p. 238) states that the path of the light ray will be such that

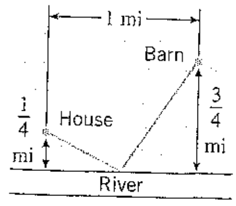
$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

where v_1 is the speed of light in the first medium, v_2 is the speed of light in the second medium, and θ_1 and θ_2 are the angles shown in Figure Ex-65. Show that this follows from the assumption that the path of minimum time occurs when $dt/dx = 0$.

66. A farmer wants to walk at a constant rate from her barn to a straight river, fill her pail, and carry it to her house in the least time.
    (a) Explain how this problem relates to Fermat's principle and the light-reflection problem in Exercise 64.
    (b) Use the result of Exercise 64 to describe geometrically the best path for the farmer to take.
    (c) Use part (b) to determine where the farmer should fill her pail if her house and barn are located as in Figure Ex-66.



▲ **Figure Ex-65**



▲ **Figure Ex-66**

67. If an unknown physical quantity x is measured n times, the measurements $x_1, x_2, \ldots, x_n$ often vary because of uncontrollable factors such as temperature, atmospheric pressure, and so forth. Thus, a scientist is often faced with the problem of using n different observed measurements to obtain an estimate $\bar{x}$ of an unknown quantity x. One method for making such an estimate is based on the ***least squares principle***, which states that the estimate $\bar{x}$ should be chosen to minimize

$$s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2$$

which is the sum of the squares of the deviations between the estimate $\bar{x}$ and the measured values. Show that the estimate resulting from the least squares principle is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

that is, $\bar{x}$ is the arithmetic average of the observed values.

68. Prove: If $f(x) \geq 0$ on an interval and if $f(x)$ has a maximum value on that interval at x_0, then $\sqrt{f(x)}$ also has a maximum value at x_0. Similarly for minimum values. [*Hint:* Use the fact that $\sqrt{x}$ is an increasing function on the interval $[0, +\infty)$.]

69. Writing Discuss the importance of finding intervals of possible values imposed by physical restrictions on variables in an applied maximum or minimum problem.

23. The *mechanic's rule* for approximating square roots states that $\sqrt{a} \approx x_{n+1}$, where

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3, \ldots$$

and x_1 is any positive approximation to $\sqrt{a}$.
(a) Apply Newton's Method to
$$f(x) = x^2 - a$$
to derive the mechanic's rule.
(b) Use the mechanic's rule to approximate $\sqrt{10}$.

24. Many calculators compute reciprocals using the approximation $1/a \approx x_{n+1}$, where

$$x_{n+1} = x_n(2 - ax_n), \quad n = 1, 2, 3, \ldots$$

and x_1 is an initial approximation to $1/a$. This formula makes it possible to perform divisions using multiplications and subtractions, which is a faster procedure than dividing directly.
(a) Apply Newton's Method to
$$f(x) = \frac{1}{x} - a$$
to derive this approximation.
(b) Use the formula to approximate $\frac{1}{17}$.

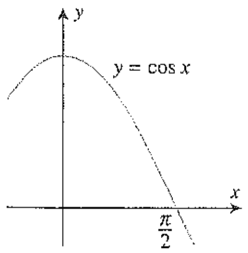
25. Use Newton's Method to approximate the absolute minimum of $f(x) = \frac{1}{4}x^4 + x^2 - 5x$.

26. Use Newton's Method to approximate the absolute maximum of $f(x) = x \sin x$ on the interval $[0, \pi]$.

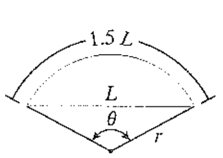
27. Use Newton's Method to approximate the coordinates of the the point on the parabola $y = x^2$ that is closest to the point $(1, 0)$.

28. Use Newton's Method to approximate the dimensions of the rectangle of largest area that can be inscribed under the curve $y = \cos x$ for $0 \le x \le \pi/2$ (Figure Ex-28).

29. (a) Show that on a circle of radius r, the central angle θ that subtends an arc whose length is 1.5 times the length L of its chord satisfies the equation $\theta = 3\sin(\theta/2)$ (Figure Ex-29).
(b) Use Newton's Method to approximate θ.

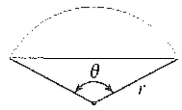


▲ Figure Ex-28



▲ Figure Ex-29

30. A *segment* of a circle is the region enclosed by an arc and its chord (Figure Ex-34). If r is the radius of the circle and θ the angle subtended at the center of the circle, then it can be shown that the area A of the segment is $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians. Find the value of θ for which the area of the segment is one-fourth the area of the circle. Give θ to the nearest degree.



◄ Figure Ex-30

31–32 Use Newton's Method to approximate all real values of y satisfying the given equation for the indicated value of x.

31. $xy^4 + x^3y = 1;\ x = 1$
32. $xy - \cos\left(\frac{1}{2}xy\right) = 0;\ x = 2$

33. An *annuity* is a sequence of equal payments that are paid or received at regular time intervals. For example, you may want to deposit equal amounts at the end of each year into an interest-bearing account for the purpose of accumulating a lump sum at some future time. If, at the end of each year, interest of $i \times 100\%$ on the account balance for that year is added to the account, then the account is said to pay $i \times 100\%$ interest, ***compounded annually***. It can be shown that if payments of Q dollars are deposited at the end of each year into an account that pays $i \times 100\%$ compounded annually, then at the time when the nth payment and the accrued interest for the past year are deposited, the amount $S(n)$ in the account is given by the formula

$$S(n) = \frac{Q}{i}[(1+i)^n - 1]$$

Suppose that you can invest \$5000 in an interest-bearing account at the end of each year, and your objective is to have \$250,000 on the 25th payment. Approximately what annual compound interest rate must the account pay for you to achieve your goal? [*Hint:* Show that the interest rate i satisfies the equation $50i = (1+i)^{25} - 1$, and solve it using Newton's Method.]

FOCUS ON CONCEPTS

34. (a) Use a graphing utility to generate the graph of
$$f(x) = \frac{x}{x^2 + 1}$$
and use it to explain what happens if you apply Newton's Method with a starting value of $x_1 = 2$. Check your conclusion by computing x_2, x_3, x_4, and x_5.
(b) Use the graph generated in part (a) to explain what happens if you apply Newton's Method with a starting value of $x_1 = 0.5$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

35. (a) Apply Newton's Method to $f(x) = x^2 + 1$ with a starting value of $x_1 = 0.5$, and determine if the values of $x_2, \ldots, x_{10}$ appear to converge.
(b) Explain what is happening.

36. In each part, explain what happens if you apply Newton's Method to a function f when the given condition is satisfied for some value of n.
(a) $f(x_n) = 0$
(b) $x_{n+1} = x_n$
(c) $x_{n+2} = x_n \neq x_{n+1}$

14. One application of the Mean-Value Theorem is to prove that a function with positive derivative on an interval must be increasing on that interval.

FOCUS ON CONCEPTS

15. Let $f(x) = \tan x$.
(a) Show that there is no point c in the interval $(0, \pi)$ such that $f'(c) = 0$, even though $f(0) = f(\pi) = 0$.
(b) Explain why the result in part (a) does not contradict Rolle's Theorem.

16. Let $f(x) = x^{2/3}$, $a = -1$, and $b = 8$.
(a) Show that there is no point c in (a, b) such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
(b) Explain why the result in part (a) does not contradict the Mean-Value Theorem.

17. (a) Show that if f is differentiable on $(-\infty, +\infty)$, and if $y = f(x)$ and $y = f'(x)$ are graphed in the same coordinate system, then between any two x-intercepts of f there is at least one x-intercept of f'.
(b) Give some examples that illustrate this.

18. Review Formulas (8) and (9) in Section 2.1 and use the Mean-Value Theorem to show that if f is differentiable on $(-\infty, +\infty)$, then for any interval $[x_0, x_1]$ there is at least one point in (x_0, x_1) where the instantaneous rate of change of y with respect to x is equal to the average rate of change over the interval.

19–21 Use the result of Exercise 18 in these exercises.

19. An automobile travels 4 mi along a straight road in 5 min. Show that the speedometer reads exactly 48 mi/h at least once during the trip.

20. At 11 A.M. on a certain morning the outside temperature was 76°F. At 11 P.M. that evening it had dropped to 52°F.
(a) Show that at some instant during this period the temperature was decreasing at the rate of 2°F/h.
(b) Suppose that you know the temperature reached a high of 88°F sometime between 11 A.M. and 11 P.M. Show that at some instant during this period the temperature was decreasing at a rate greater than 3°F/h.

21. Suppose that two runners in a 100 m dash finish in a tie. Show that they had the same velocity at least once during the race.

22. Use the fact that
$$\frac{d}{dx}(3x^4 + x^2 - 4x) = 12x^3 + 2x - 4$$
to show that the equation $12x^3 + 2x - 4 = 0$ has at least one solution in the interval $(0, 1)$.

23. (a) Use the Constant Difference Theorem (3.8.3) to show that if $f'(x) = g'(x)$ for all x in the interval $(-\infty, +\infty)$, and if f and g have the same value at some point x_0, then $f(x) = g(x)$ for all x in $(-\infty, +\infty)$.
(b) Use the result in part (a) to confirm the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

24. (a) Use the Constant Difference Theorem (3.8.3) to show that if $f'(x) = g'(x)$ for all x in $(-\infty, +\infty)$, and if $f(x_0) - g(x_0) = c$ at some point x_0, then
$$f(x) - g(x) = c$$
for all x in $(-\infty, +\infty)$.
(b) Use the result in part (a) to show that the function
$$h(x) = (x - 1)^3 - (x^2 + 3)(x - 3)$$
is constant for all x in $(-\infty, +\infty)$, and find the constant.
(c) Check the result in part (b) by multiplying out and simplifying the formula for $h(x)$.

FOCUS ON CONCEPTS

25. (a) Use the Mean-Value Theorem to show that if f is differentiable on an interval, and if $|f'(x)| \le M$ for all values of x in the interval, then
$$|f(x) - f(y)| \le M|x - y|$$
for all values of x and y in the interval.
(b) Use the result in part (a) to show that
$$|\sin x - \sin y| \le |x - y|$$
for all real values of x and y.

26. (a) Use the Mean-Value Theorem to show that if f is differentiable on an open interval, and if $|f'(x)| \ge M$ for all values of x in the interval, then
$$|f(x) - f(y)| \ge M|x - y|$$
for all values of x and y in the interval.
(b) Use the result in part (a) to show that
$$|\tan x - \tan y| \ge |x - y|$$
for all values of x and y in the interval $(-\pi/2, \pi/2)$.
(c) Use the result in part (b) to show that
$$|\tan x + \tan y| \ge |x + y|$$
for all values of x and y in the interval $(-\pi/2, \pi/2)$.

27. (a) Use the Mean-Value Theorem to show that
$$\sqrt{y} - \sqrt{x} < \frac{y - x}{2\sqrt{x}}$$
if $0 < x < y$.
(b) Use the result in part (a) to show that if $0 < x < y$, then $\sqrt{xy} < \frac{1}{2}(x + y)$.

28. Show that if f is differentiable on an open interval and $f'(x) \ne 0$ on the interval, the equation $f(x) = 0$ can have at most one real root in the interval.

29. Use the result in Exercise 28 to show the following:
(a) The equation $x^3 + 4x - 1 = 0$ has exactly one real root.
(b) If $b^2 - 3ac < 0$ and if $a \ne 0$, then the equation
$$ax^3 + bx^2 + cx + d = 0$$
has exactly one real root.