Assignment 3 MA 1125

Q 1 From the definition of derivative prove $\frac{d}{dx}(\cos x) = -\sin x$.

Q 2. Find dy/dx if

(i) $y = \frac{\sec x}{1 + \tan x}$

(ii) $y = x^3 \sin x \cos x$.

(iii) $y = \frac{x^2 + 1}{(x^3 - 1)(x^4 - 7x + 2)}$

From the book - Anton

p 141 # 68 p 121 # 26

p 142 # 72

p 151 # 32, 22, 18.

These pages from the book follow.

1–18 Find $f'(x)$.

1. $f(x) = 4\cos x + 2\sin x$
2. $f(x) = \dfrac{5}{x^2} + \sin x$
3. $f(x) = -4x^2 \cos x$
4. $f(x) = 2\sin^2 x$
5. $f(x) = \dfrac{5 - \cos x}{5 + \sin x}$
6. $f(x) = \dfrac{\sin x}{x^2 + \sin x}$
7. $f(x) = \sec x - \sqrt{2}\tan x$
8. $f(x) = (x^2 + 1)\sec x$
9. $f(x) = 4\csc x - \cot x$
10. $f(x) = \cos x - x\csc x$
11. $f(x) = \sec x \tan x$
12. $f(x) = \csc x \cot x$
13. $f(x) = \dfrac{\cot x}{1 + \csc x}$
14. $f(x) = \dfrac{\sec x}{1 + \tan x}$
15. $f(x) = \sin^2 x + \cos^2 x$
16. $f(x) = \sec^2 x - \tan^2 x$
17. $f(x) = \dfrac{\sin x \sec x}{1 + x\tan x}$
18. $f(x) = \dfrac{(x^2 + 1)\cot x}{3 - \cos x \csc x}$

19–24 Find d^2y/dx^2.

19. $y = x\cos x$
20. $y = \csc x$
21. $y = x\sin x - 3\cos x$
22. $y = x^2 \cos x + 4\sin x$
23. $y = \sin x \cos x$
24. $y = \tan x$

25. Find the equation of the line tangent to the graph of $\tan x$ at
    (a) $x = 0$ (b) $x = \pi/4$ (c) $x = -\pi/4$.
26. Find the equation of the line tangent to the graph of $\sin x$ at
    (a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/4$.
27. (a) Show that $y = x\sin x$ is a solution to $y'' + y = 2\cos x$.
    (b) Show that $y = x\sin x$ is a solution of the equation $y^{(4)} + y'' = -2\cos x$.
28. (a) Show that $y = \cos x$ and $y = \sin x$ are solutions of the equation $y'' + y = 0$.
    (b) Show that $y = A\sin x + B\cos x$ is a solution of the equation $y'' + y = 0$ for all constants A and B.
29. Find all values in the interval $[-2\pi, 2\pi]$ at which the graph of f has a horizontal tangent line.
    (a) $f(x) = \sin x$ (b) $f(x) = x + \cos x$
    (c) $f(x) = \tan x$ (d) $f(x) = \sec x$
30. (a) Use a graphing utility to make rough estimates of the values in the interval $[0, 2\pi]$ at which the graph of $y = \sin x \cos x$ has a horizontal tangent line.
    (b) Find the exact locations of the points where the graph has a horizontal tangent line.

31. A 10 ft ladder leans against a wall at an angle θ with the horizontal, as shown in the accompanying figure. The top of the ladder is x feet above the ground. If the bottom of the ladder is pushed toward the wall, find the rate at which x changes with respect to θ when $\theta = 60°$. Express the answer in units of feet/degree.

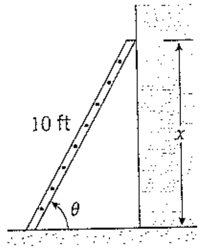

Figure Ex-31

32. An airplane is flying on a horizontal path at a height of 3800 ft, as shown in the accompanying figure. At what rate is the distance s between the airplane and the fixed point P changing with respect to θ when $\theta = 30°$? Express the answer in units of feet/degree.

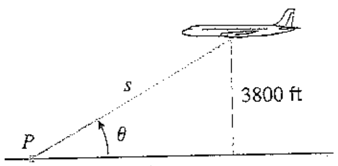

Figure Ex-32

33. A searchlight is trained on the side of a tall building. As the light rotates, the spot it illuminates moves up and down the side of the building. That is, the distance D between ground level and the illuminated spot on the side of the building is a function of the angle θ formed by the light beam and the horizontal (see the accompanying figure). If the searchlight is located 50 m from the building, find the rate at which D is changing with respect to θ when $\theta = 45°$. Express your answer in units of meters/degree.

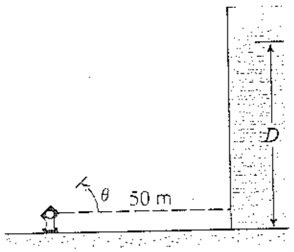

Figure Ex-33

70. In each part, compute f', f'', f''', and then state the formula for $f^{(n)}$.
(a) $f(x) = 1/x$ (b) $f(x) = 1/x^2$
[*Hint:* The expression $(-1)^n$ has a value of 1 if n is even and -1 if n is odd. Use this expression in your answer.]

71. (a) Prove:

$$\frac{d^2}{dx^2}[cf(x)] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx^2}[f(x)+g(x)] = \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) Do the results in part (a) generalize to nth derivatives? Justify your answer.

72. Let $f(x) = x^8 - 2x + 3$; find

$$\lim_{w \to 2} \frac{f'(w) - f'(2)}{w - 2}$$

73. (a) Find $f^{(n)}(x)$ if $f(x) = x^n$, $n = 1, 2, 3, \ldots$.
(b) Find $f^{(n)}(x)$ if $f(x) = x^k$ and $n > k$, where k is a positive integer.
(c) Find $f^{(n)}(x)$ if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

74. (a) Prove: If $f''(x)$ exists for each x in (a, b), then both f and f' are continuous on (a, b).
(b) What can be said about the continuity of f and its derivatives if $f^{(n)}(x)$ exists for each x in (a, b)?

75. Let $f(x) = (mx + b)^n$, where m and b are constants and n is an integer. Use the result of Exercise 52 in Section 2.2 to prove that $f'(x) = nm(mx + b)^{n-1}$.

76–77 Verify the result of Exercise 75 for $f(x)$.

76. $f(x) = (2x + 3)^2$ **77.** $f(x) = (3x - 1)^3$

78–81 Use the result of Exercise 75 to compute the derivative of the given function $f(x)$.

78. $f(x) = \dfrac{1}{x - 1}$

79. $f(x) = \dfrac{3}{(2x + 1)^2}$

80. $f(x) = \dfrac{x}{x + 1}$

81. $f(x) = \dfrac{2x^2 + 4x + 3}{x^2 + 2x + 1}$

82. The purpose of this exercise is to extend the power rule (Theorem 2.3.2) to any integer exponent. Let $f(x) = x^n$, where n is any integer. If $n > 0$, then $f'(x) = nx^{n-1}$ by Theorem 2.3.2.
(a) Show that the conclusion of Theorem 2.3.2 holds in the case $n = 0$.
(b) Suppose that $n < 0$ and set $m = -n$ so that

$$f(x) = x^n = x^{-m} = \frac{1}{x^m}$$

Use Definition 2.2.1 and Theorem 2.3.2 to show that

$$\frac{d}{dx}\left[\frac{1}{x^m}\right] = -mx^{m-1} \cdot \frac{1}{x^{2m}}$$

and conclude that $f'(x) = nx^{n-1}$.

QUICK CHECK ANSWERS 2.3

1. (a) 0 (b) $\sqrt{6}$ (c) $3/\sqrt{x}$ (d) $\sqrt{6}/(2\sqrt{x})$ 2. (a) $3x^2$ (b) $5x^4 + 10x$ (c) $\frac{3}{2}x^2$ (d) $1 - 10x^{-3}$ 3. 6 4. $18x - 6$

2.4 THE PRODUCT AND QUOTIENT RULES

In this section we will develop techniques for differentiating products and quotients of functions whose derivatives are known.

FOCUS ON CONCEPTS

51. Find a function $y = ax^2 + bx + c$ whose graph has an x-intercept of 1, a y-intercept of -2, and a tangent line with a slope of -1 at the y-intercept.

52. Find k if the curve $y = x^2 + k$ is tangent to the line $y = 2x$.

53. Find the x-coordinate of the point on the graph of $y = x^2$ where the tangent line is parallel to the secant line that cuts the curve at $x = -1$ and $x = 2$.

54. Find the x-coordinate of the point on the graph of $y = \sqrt{x}$ where the tangent line is parallel to the secant line that cuts the curve at $x = 1$ and $x = 4$.

55. Find the coordinates of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point $(2, 0)$.

56. Show that any two tangent lines to the parabola $y = ax^2$, $a \neq 0$, intersect at a point that is on the vertical line halfway between the points of tangency.

57. Suppose that L is the tangent line at $x = x_0$ to the graph of the cubic equation $y = ax^3 + bx$. Find the x-coordinate of the point where L intersects the graph a second time.

58. Show that the segment of the tangent line to the graph of $y = 1/x$ that is cut off by the coordinate axes is bisected by the point of tangency.

59. Show that the triangle that is formed by any tangent line to the graph of $y = 1/x$, $x > 0$, and the coordinate axes has an area of 2 square units.

60. Find conditions on a, b, c, and d so that the graph of the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has
(a) exactly two horizontal tangents
(b) exactly one horizontal tangent
(c) no horizontal tangents.

61. Newton's Law of Universal Gravitation states that the magnitude F of the force exerted by a point with mass M on a

changes? [*Hint:* Consider the size of dR/dT in the interval $0 \le T \le 700$.]

63–64 Use a graphing utility to make rough estimates of the intervals on which $f'(x) > 0$, and then find those intervals exactly by differentiating.

63. $f(x) = x - \dfrac{1}{x}$

64. $f(x) = x^3 - 3x$

65–68 You are asked in these exercises to determine whether a piecewise-defined function f is differentiable at a value $x = x_0$, where f is defined by different formulas on different sides of x_0. You may use without proof the following result, which is a consequence of the Mean-Value Theorem (discussed in Section 3.8). ***Theorem.*** *Let f be continuous at x_0 and suppose that $\lim_{x\to x_0} f'(x)$ exists. Then f is differentiable at x_0, and $f'(x_0) = \lim_{x\to x_0} f'(x)$.*

65. Show that
$$f(x) = \begin{cases} x^2 + x + 1, & x \le 1 \\ 3x, & x > 1 \end{cases}$$
is continuous at $x = 1$. Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there. Sketch the graph of f.

66. Let
$$f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ \sqrt{x}, & x \ge 9 \end{cases}$$
Is f continuous at $x = 9$? Determinc whether f is differentiable at $x = 9$. If so, find the value of the derivative there.

67. Let
$$f(x) = \begin{cases} x^2, & x \le 1 \\ \sqrt{x}, & x > 1 \end{cases}$$
Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there.

68. Let
$$f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < \frac{1}{2} \\ \frac{3}{4}x^2, & x \ge \frac{1}{2} \end{cases}$$
Determine whether f is differentiable at $x = \frac{1}{2}$. If so, find the value of the derivative there.

69. Find all points where f fails to be differentiable. Justify your answer.
(a) $f(x) = |3x - 2|$ (b) $f(x) = |x^2 - 4|$

(a) Estimate the maximum temperature and the time at which it occurs.
(b) The temperature rise is fairly linear from 8 A.M. to 2 P.M. Estimate the rate at which the temperature is increasing during this time period.
(c) Estimate the time at which the temperature is decreasing most rapidly. Estimate the instantaneous rate of change of temperature with respect to time at this instant.

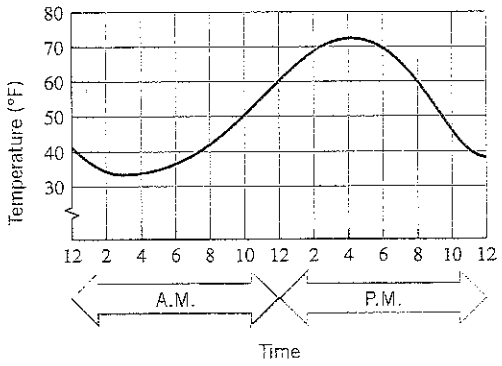

Figure Ex-23

24. The accompanying figure shows the graph of the pressure p in atmospheres (atm) versus the volume V in liters (L) of 1 mole of an ideal gas at a constant temperature of 300 K (kelvins). Use the line segments shown in the figure to estimate the rate of change of pressure with respect to volume at the points where $V = 10$ L and $V = 25$ L.

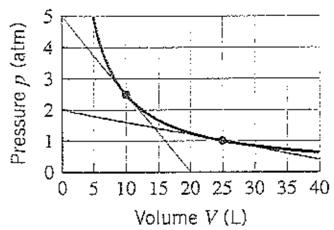

Figure Ex-24

25. The accompanying figure shows the graph of the height h in centimeters versus the age t in years of an individual from birth to age 20.

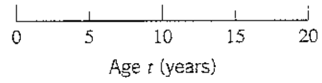

Figure Ex-25

26. An object is released from rest (its initial velocity is zero) from the Empire State Building at a height of 1250 ft above street level (Figure Ex-26). The height of the object can be modeled by the position function $s = f(t) = 1250 - 16t^2$.
(a) Verify that the object is still falling at $t = 5$ s.
(b) Find the average velocity of the object over the time interval from $t = 5$ to $t = 6$ s.
(c) Find the object's instantaneous velocity at time $t = 5$ s.

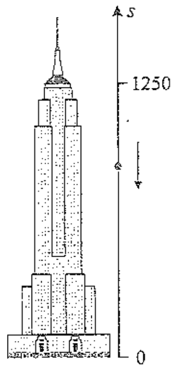

Figure Ex-26

27. During the first 40 s of a rocket flight, the rocket is propelled straight up so that in t seconds it reaches a height of $s = 0.3t^3$ ft.
(a) How high does the rocket travel in 40 s?
(b) What is the average velocity of the rocket during the first 40 s?
(c) What is the average velocity of the rocket during the first 1000 ft of its flight?
(d) What is the instantaneous velocity of the rocket at the end of 40 s?

28. An automobile is driven down a straight highway such that after $0 \le t \le 12$ seconds it is $s = 4.5t^2$ feet from its initial position.

(cont.)