1. (a) Use the ϵ, δ definition of limit to evaluate $\lim_{x\to 1} 2x^2 + 3x - 2$.

   (b) If $f(x) = \frac{9x-3\sin 3x}{5x^2}$ for$x \neq 0$ and $=$ c for x $=$ 0. For what value of c is f(x) continuous at x $=$ 0?

   (c) Find $\lim_{x\to 0^+} x \ln x$

   (d) State the squeezing theorem for limits and use it to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$

2. (a) Derive the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for Newton's Method. Does Newton's Method always work?

   (b) Find $\frac{dy}{dx}$ if

      i. $y = \sqrt{\tan(\exp(x^3 + x))}$

      ii. $x^2y^2 + x^2 \exp y = \ln(xy^3)$

   (c) A girl flies a kite at a constant height of 30 metres. The wind carries the kite horizontally at 40 m/sec. How fast must she let the string out when the kite is 50 metres away from her?

   (d) Find where the function $y = x^4 - 4x^3$ is increasing, decreasing, concave up, concave down, has local extrema, and points of inflection. Draw a rough sketch of the function.

3. (a) State The Intermediate Value Theorem. State the Extreme Value Theorem.

   (b) Solve the following

      i. $x\frac{dy}{dx} + y = x^2$.

      ii. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6 = 0$

      iii. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6 = x^2$

   (c) You are planning to close off a corner of the first quadrant in the plane with a line segment 20 units long running from (a,0) to (0,b). What is the largest area possible?

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4. (a) How is the Riemann Integral $\int_a^b f(x)dx$ defined?

   (b) Integrate the following.

      i. $\int x \ln(x^2+3)dx$

      ii. $\int x \sin x dx$

      iii. $\displaystyle\int \frac{x^2+2}{(x+2)(x+3)}$

      iv. $\displaystyle\int \frac{x}{x^2+x+1}dx$

5. (a) Derive the formula for the length of the curve y = f(x) between x = a and x = b.

   (b) Find the area enclosed between $y = x^3 - 3x^2$ and the x-axis.

   (c) The bounded region between $y = x$, $y = -x+2$ and above the x-axis is rotated about the y-axis. Find the volume first using washers and then using cylindrical shells.

6. (a) Define $\sum_{n=1}^{\infty} a_n = L$.

   (b) Do the following series converge or diverge? Give reasons.

      i. $\displaystyle\sum_{n=1}^{\infty} \frac{n^7}{7^n}$

      ii. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n}$

      iii. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

   (c) Find the Taylor Series for $f(x) = \exp(x)$ about x = 0.

   (d) Find for what values of x the following power series converges absolutely, conditionally, or diverges? What is the radius of convergence?
   $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n^3}$

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1 (a) $\lim_{x \to 1} 2x^2+3x-2 = 3$

$|2x^2+3x-2-3| = |2x^2+3x-5|$

$= |2x+5||x-1|$

if $|x-1|<1$ $\quad 0<x<2$.

$|2x+5| < 9$.

Let $\delta = \min(\epsilon/9, 1)$

then $|2x^2+3x-2-3| < 9.\delta \leq \epsilon$.

(b) $f(x) = \dfrac{9x - 3\sin 3x}{5x^2}$ $\quad x \neq 0$.

$x \to 0 \quad \to \dfrac{0}{0}$

$\overset{L}{=} \dfrac{9-9\cos x}{10x}$ L'Hospital.

$\overset{L}{=} \dfrac{9\sin x}{10} \to 0$. L'Hospital

So for $f(x)$ to be continuous at 0 we need $f(0)=0$

i.e. $c=0$

1 (c) $\lim_{x \to 0^+} x \cdot \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \to \frac{-\infty}{\infty}$

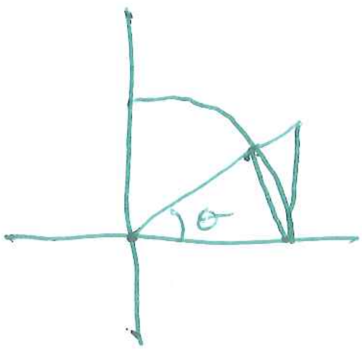
$$\mathcal{L} \quad \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\frac{x^2}{x} = -x \to 0.$$

(d) Squeezing Theorem

if $f(x) \leq g(x) \leq h(x)$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L$



Area of small Δ

$= \frac{1}{2} \cdot \sin\theta.$

Area of sector

$= \frac{1}{2} \cdot \theta.$

Area of large Δ

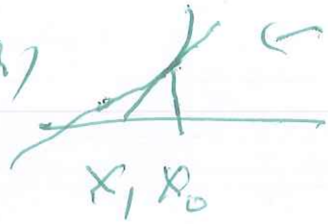
$= \frac{1}{2} \tan\theta.$

So $\sin\theta \leq \theta \leq \frac{\sin\theta}{\cos\theta}.$

So $1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}.$

as $\theta \to 0$ $\frac{1}{\cos\theta} \to 1$ $\therefore \lim \frac{\theta}{\sin\theta} = 1$

$\therefore \lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$

2. (a) 

← equation of tangent line at $(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

x_1 occurs when $y = 0$.

$$-f(x_0) = f'(x_0)(x_1 - x_0)$$

$$f'(x_0).x_0 - f(x_0) = f'(x_0).x_1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Sim., $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Clearly this breaks down if $f'(x_n) = 0$. But even without this then x_n do not always tend to a root. They may oscillate or even go off to ∞.

(b) $y = \sqrt{\tan(\exp(x^3 + x))}$

(i) $$\frac{dy}{dx} = \frac{1}{2}(\tan(\exp(x^3+x)))^{-1/2}.\sec^2(\exp(x^3+x)).$$
$$.\exp(x^3+x).(3x^2+1).$$

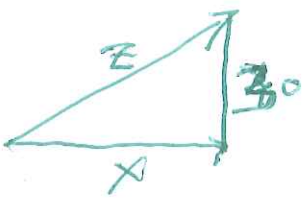
2. (ii) $x^2y^2 + x^2 \exp y = \ln(xy^3).$

$$2xy^2 + x^2 2y\frac{dy}{dx} + 2x\exp y + x^2e^y\frac{dy}{dx}$$

$$= \frac{1}{xy^3}\left(y^3 + x3y^2\frac{dy}{dx}\right).$$

$$\frac{dy}{dx} = \frac{\frac{1}{x} - 2xy^2 - 2x\exp y}{+2x^2y + x^2\exp y - \frac{3}{y}}$$

(c)



$$z^2 = x^2 + 30^2.$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt}.$$

when $z = 50$ $x = 40$

$$\frac{dx}{dt} = 40.$$

$$50.\frac{dz}{dt} = 40.40.$$

$$\frac{dz}{dt} = 32 \text{ m/sec}.$$

2(d). $\sum \frac{n^7}{7^n}$

$\frac{(n+1)^7}{7^{n+1}} \cdot \frac{7^n}{n^7} \to \frac{1}{7} <$

$\cancel{(-1)^n \frac{\ln n}{n}}$

$\cancel{\ln n}$

$y = x^4 - 4x^3$

$= x^3(x-4)$.

$\frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x-3)$

$x=0, \; x=3$.

>0 on $x>3$ $f(x) \uparrow$

<0 on $x<3$. $f(x) \downarrow$ $\boxed{x=3 \text{ loc min.}}$

$\frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x-2)$.

$=0$ at $x=0$, $x=2$

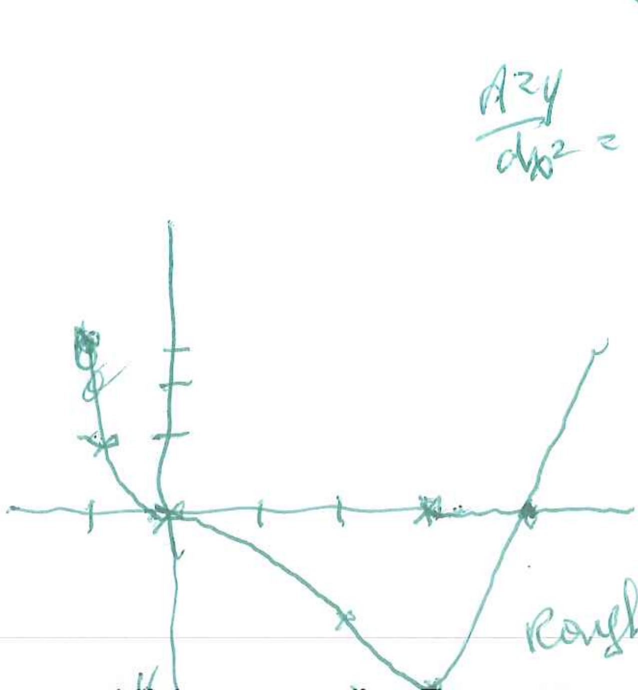
pts of inflection

>0 at $x>2$, $x<0$.

Concave up

<0 at $(0,2)$.

concave down



Rough.

3. (a) Let $f(x)$ be cont. on $[a, b]$. I.V.T.

Then if l lies between $f(a)$ and $f(b)$, $\exists$ c in $[a, b]$ such that $f(c) = l$.

E.V.T. Let $f(x)$ be cont. on $[a, b]$ then $\exists$ c_1, c_2 in $[a, b]$ such that

$$f(c_1) = \max_{x \in [a,b]} f(x) \qquad f(c_2) = \min_{x \in [a,b]} f(x).$$

(b) (i)

$$x \frac{dy}{dx} + y = x^2$$

$$\frac{dy}{dx} + \frac{1}{x} y = x.$$

$$P(x) = \frac{1}{x} \qquad \int P(x) = \ln x$$

$$e^{\int P(x)} = e^{\ln x} = x.$$

$$x \frac{dy}{dx} + y = x^2$$

$$\frac{d(xy)}{dx} = x^2$$

$$xy = \frac{x^3}{3} + c.$$

$$y = \frac{x^2}{3} + \frac{c}{x}.$$

3(b)(ii) $(D^2+5D)y=-6$.

$D(D+5)Dy = 0$

Solutions $y = A_1 e^{-5x} + A_2 + A_3 x$.

$(D^2+5D)(A_1 e^{-5x} + A_2 + A_3 x)$

$= (D^2+5D)(A_3 x) = 5A_3 = -6$

$A_3 = -6/5$.

Solution $y = A_1 e^{-5x} + A_2 - 6/5\, x$.

or

$(D^2+5D+6)y = 0$

$y = A_1 e^{-2x} + A_2 e^{-3x}$

(iii) $(D^2+5D)y = x^2-6$.

$D^3(D^2+5D)y = 0$.

$y = A_1 e^{-5x} + A_2 + A_3 x + A_4 x^2 + A_5 x^3$

$D(D+5)(A_1 e^{-5x} + A_2 + A_3 x + A_4 x^2 + A_5 x^3)$

$= D(D+5)(A_5 x^3 + A_4 x^2 + A_3 x)$.

$D^2(A_5 x^3 + A_4 x^2 + A_3 x) = 6A_5 x + 2A_4$.

$5D(A_5 x^3 + A_4 x^2 + A_3 x) = 15 A_5 x^2 + 10 A_4 x + 5A_3$

$= x^2 - 6$.

$$\boxed{\begin{array}{l} A_3 = -\cancel{6/5} \quad -\frac{148}{125} \\ A_5 = 1/15 \quad A_4 = -\frac{1}{25} \end{array}}$$

$15A_5 = 1$, $10A_4 + 6A_5 = 0$.

$5A_3 = -6$

3 (b) (iii) or

$(D+2)(D+3)y = x^2$

$D^3(D+2)(D+3)y = 0$

$y = A_1 e^{-2x} + A_2 e^{-3x} + A_3 + A_4 x + A_5 x^2$

$(D+2)(D+3)y = (D+2)(D+3)(A_3 + A_4 x + A_5 x^2)$

$y = A_3 + A_4 x + A_5 x^2$

$Dy = A_4 + 2A_5 x$

$D^2 y = 2A_5$

$D^2 y + 5Dy + 6y = 2A_5 + 5A_4 + 6A_3$

$+ 10A_5 x + 6A_4 x$

$+ 6A_5 x^2 = x^2$

$A_5 = \frac{1}{6}$

$\frac{10}{6} + 6A_4 = 0$

$A_4 = -\frac{10}{36}$

$6A_3 = -\frac{2}{6} + \frac{50}{36}$

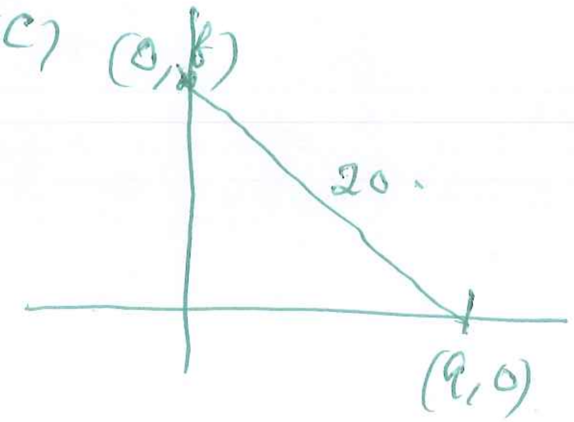
$= \frac{38}{36}$

$A_3 = \frac{38}{216}$

Soln $y = A_1 e^{-2x} + A_2 e^{-3x} + \frac{38}{216} - \frac{10}{36}x + \frac{1}{6}x^2$

$\frac{19}{108}$ $\frac{5}{18}$

3.(c)



Area $= \frac{1}{2}ab$. $20 = \sqrt{a^2+b^2}$.

$a = \sqrt{400-b^2}$.

$A = \frac{1}{2}b \cdot \sqrt{400-b^2}$.

$\frac{dA}{db} = \frac{1}{2}\sqrt{400-b^2} + \frac{1}{4}b \frac{1}{\sqrt{400-b^2}} \cdot -2b. = 0.$

$\sqrt{400-b^2} = + \frac{b^2}{\sqrt{400-b^2}}$

$400 - b^2 = b^2$

$400 = 2b^2$

$b = 10\sqrt{2}$. $a = 10\sqrt{2}$

4 (a) Let P be any partition on $[a, b]$

$P = \{x_0, x_1, \dots, x_n\}$ with

$$a = x_0 < x_1 < x_2 \dots < x_n = b$$

then choose any x_i^* in $[x_{i-1}, x_i]$.

Now form the Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

Let $\|P\| = \max \Delta x_i$

Now look for $\lim\limits_{\|P\| \to 0} \sum f(x_i^*) \Delta x_i$

<u>if</u> this limit exists, this is

$$\int_a^b f(x)\,dx.$$

4 (b) (i) $\int x \ln(x^2+3)\,dx$

$u = x^2+3$

$\frac{du}{dx} = 2x$

$\int \frac{1}{2}\frac{du}{dx} \ln u \,dx = \frac{1}{2}\int \ln u \,du$

$=$ $f = \ln u$ $\quad \frac{dg}{du} = 1$

$\frac{df}{dx} = \frac{1}{u}$ $\quad g = u$

$u \ln u = \int \ln u + \int 1\,du$

$= \int \ln u + u$

$= \frac{1}{2}(u \ln u - u)$

$= \frac{1}{2}[(x^2+3)\ln(x^2+3) - (x^2+3)] + C$

(ii) $\int x \sin x \,dx$

$f = x \quad \frac{dg}{dx} = \sin x$

$\frac{df}{dx} = 1 \quad g = -\cos x$

$-x\cos x = \int x \sin x\,dx + \int -\cos x\,dx$

$= \int x \sin x \,\cancel{dx} - \sin x$

$\int x \sin x \,dx = \sin x - x\cos x + C$

(iii)

$$\int \frac{x^2+2}{x^2+5x+6}\,dx$$

Since the degree of the numerator $\geq$ degree of the denominator we must first divide

$$\begin{array}{r|l} & 1 \\ x^2+5x+6 & x^2+2 \\ & x^2+5x+6 \\ \hline & -5x-4 \end{array}$$

$$\frac{x^2+2}{x^2+5x+6} = 1 - \frac{5x+4}{(x+2)(x+3)}$$

$$\frac{5x+4}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$5x+4 = A(x+3) + B(x+2)$$

$x=-3 \quad -11 = -B \qquad B = 11$

$x=-2 \quad -6 = A \qquad A = -6$

$$\int \frac{x^2+2}{(x+2)(x+3)}\,dx = \int 1\,dx + \int \frac{6}{x+2}\,dx - \int \frac{11}{x+3}\,dx$$

$$= x + 6\ln|x+2| - 11\ln|x+3| + C$$

(iv) $\int \frac{x}{x^2+x+1}\,dx$.

$$\frac{x}{x^2+x+1} = \frac{x}{(x+1/2)^2+3/4}$$

$u = x+1/2$

$\frac{du}{dx} = 1$

$$\int \frac{x}{x^2+x+1}\,dx = \int \frac{u-1/2}{u^2+3/4}\frac{du}{dx}\,dx$$

$$= \int \frac{u}{u^2+3/4}\,du - 1/2\int \frac{1}{u^2+3/4}\,du.$$

$v = u^2+3/4$

$\frac{dv}{du} = 2u$

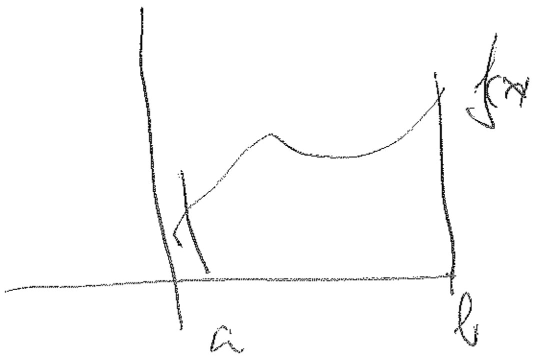
$$= 1/2\int \frac{\frac{dv}{du}}{v}\,du - 1/2\int \frac{1}{u^2+3/4}\,du$$

$$= 1/2\int \frac{1}{v}\,dv - 1/2\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(u\cdot\frac{2}{\sqrt{3}}\right)$$

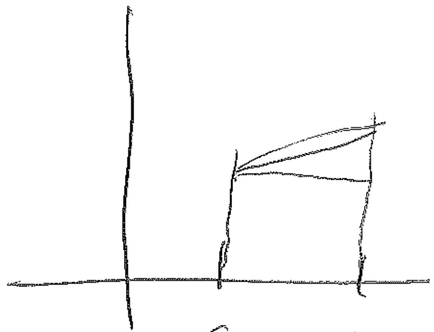
$$= 1/2\ln v - \frac{1}{\sqrt{3}}\tan^{-1}\left[(x+1/2)\frac{2}{\sqrt{3}}\right]$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

5 (a) Consider a curve $y = f(x)$



Take a small piece from x_{i-1} to x_i



approximate the curve length by the straight line length.

$\sim \sqrt{(f(x_{i-1}) - f(x_i))^2 + (x_{i-1} - x_i)^2}$.

By the mean value theorem $\exists\ x_i^*$ in $[x_{i-1}, x_i]$ with $f'(x_i^*) = \dfrac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$

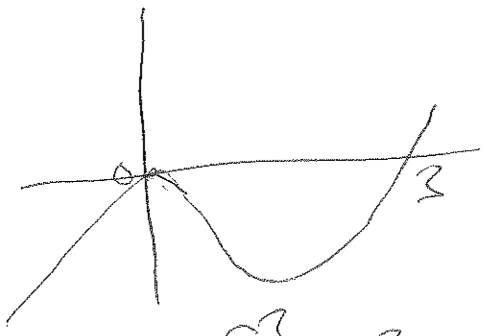
$S_i \sim \sqrt{f'(x_i^*)\Delta x_i^2 + \Delta x_i^2} = \sqrt{f'(x_i^*)^2 + 1}\ \Delta x_i$

So we get a Riemann Sum

$$\sum \sqrt{f'(x_i^*)^2 + 1}\ .\ \Delta x.$$

Let $\|P\| \to 0 \quad \longrightarrow \quad \int_a^b \sqrt{f'(x)^2 + 1}\ dx$

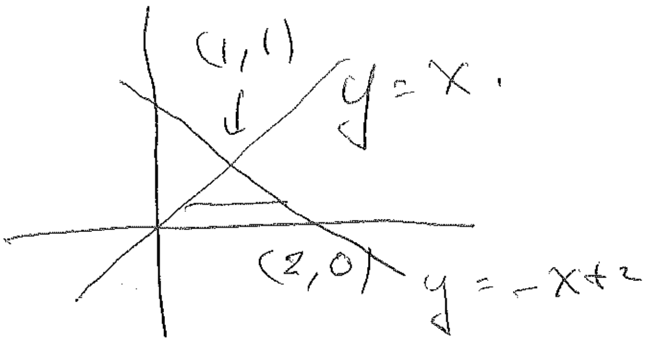
5 (b) $y = x^3 - 3x^2$ looks like



$$\int_0^3 x^3 - 3x^2\,dx = \frac{x^4}{4} - x^3 \Big|_0^3 = -6\tfrac{3}{4}$$

So area $= 6\tfrac{3}{4}$.

(c)



$x = -x+2$

$2x = 2 \quad x = 1$

$\Rightarrow y = 1$

Rotated about y-axis

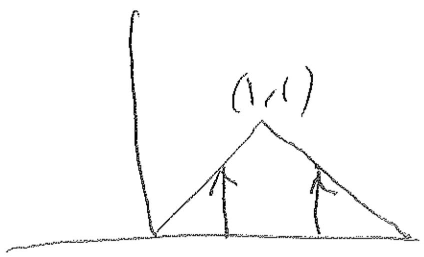
By disks Outer ⊙ $\pi \int_0^1 (2-y)^2\,dy$,

Inner ⊙ $\pi \int_0^1 y^2\,dy$,

π ~~$\frac{(2-y)}{}$~~ $-\frac{\pi(2-y)^3}{3}\Big|_0^1 - \pi\frac{y^3}{3}\Big|_0^1$

$= -\frac{\pi}{3} + \frac{\pi}{3}\cdot 8 - \pi/3 = 2\pi$

5 (c) By cylindrical slices.



$$\int_0^1 2\pi x \cdot x \cdot dx + \int_1^2 2\pi x(-x+2)\,dx$$

$$= 2\pi \frac{x^3}{3}\Big|_0^1 + \left(-2\pi\frac{x^3}{3} + 2\pi x^2\right)\Big|_1^2$$

$$= 2\pi/3 + -\frac{16\pi}{3} + 8\pi - \left(-2\pi/3 + 2\pi\right)$$

$$= 2\pi/3 - 16\pi/3 + 6\pi + 2\pi/3$$

$$= 2\pi$$

6 (a) Let $S_n = \sum_{k=1}^{n} a_k$, then $\sum_{k=1}^{\infty} a_k = L$

means $\lim_{n\to\infty} S_n = L$

(b) (i) $\sum \frac{n^7}{7^n}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{(n+1)^7}{7^{n+1}} \cdot \frac{7^n}{n^7} = \left(\frac{n+1}{n}\right)^7 \cdot \frac{1}{7}$

$\to \frac{1}{7}$ as $n \to \infty$

< 1

So Ratio Test says

$\sum \frac{n^7}{7^n}$ converges.

6 ~~(b)~~ (ii) $\sum \frac{n-2}{n}$ $\quad \frac{n-2}{n} \to 1$ as $n \to \infty$.

$\therefore \sum \frac{n-2}{n}$ diverges by Divergence Test.

~~(b)~~ (iii) $\sum \frac{(-1)^n \ln n}{n}$. This is an alternating series. $\frac{\ln n}{n} \to 0$.

and $a_{n+1} < a_n$ at least eventually, so the series converges

(c) $f(x) = e^x$. $\quad f^n(x) = e^x \quad f^n(0) = 1$.

The Taylor series at $x=0$ is

$\sum \frac{f^n(0)}{n!} \cdot x^n$ so we get

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(d) $\sum \frac{(-1)^n (x+2)^n}{n^3}$ $\quad \frac{|a_{n+1}|}{|a_n|} = \frac{|x+2|^{n+1}}{(n+1)^3} \cdot \frac{n^3}{|x+2|^n}$

$= |x+2| \cdot \left(\frac{n}{n+1}\right)^3 \longrightarrow |x+2|$

so converges absolutely if $|x+2| < 1$.

i.e. $-3 < x < -1$

if $x = -1$. $\sum \frac{(-1)^n}{n^3}$ converges absolutely.

$x = -3$ $\quad \sum \frac{1}{n^3}$ converges absolutely

So converges absolutely $-3 \leq x \leq -1$, Diverges elsewhere. Radius of convergence $= 1$