The second integral on the right now involves a proper rational function and can thus be evaluated by a partial fraction decomposition. Using the result of Example 1 we obtain

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C \blacktriangleleft$$

CONCLUDING REMARKS

There are some cases in which the method of partial fractions is inappropriate. For example, it would be inefficient to use partial fractions to perform the integration

$$\int \frac{3x^2+2}{x^3+2x-8}\,dx = \ln|x^3+2x-8|+C$$

since the substitution $u = x^3+2x-8$ is more direct. Similarly, the integration

$$\int \frac{2x-1}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} = \ln(x^2+1) - \tan^{-1}x + C$$

requires only a little algebra since the integrand is already in partial fraction form.

✔ QUICK CHECK EXERCISES 7.5 (See page 523 for answers.)

1. A partial fraction is a rational function of the form ______ or of the form ______.
2. (a) What is a proper rational function?
   (b) What condition must the degree of the numerator and the degree of the denominator of a rational function satisfy for the method of partial fractions to be applicable directly?
   (c) If the condition in part (b) is not satisfied, what must you do if you want to use partial fractions?
3. Suppose that the function $f(x) = P(x)/Q(x)$ is a proper rational function.
   (a) For each factor of $Q(x)$ of the form $(ax+b)^m$, the partial fraction decomposition of f contains the following sum of m partial fractions: ______
   (b) For each factor of $Q(x)$ of the form $(ax^2+bx+c)^m$, where ax^2+bx+c is an irreducible quadratic, the partial fraction decomposition of f contains the following sum of m partial fractions: ______.
4. Complete the partial fraction decomposition.
   (a) $\dfrac{-3}{(x+1)(2x-1)} = \dfrac{A}{x+1} - \dfrac{2}{2x-1}$
   (b) $\dfrac{2x^2-3x}{(x^2+1)(3x+2)} = \dfrac{B}{3x+2} - \dfrac{1}{x^2+1}$
5. Evaluate the integral.
   (a) $\displaystyle\int \frac{3}{(x+1)(1-2x)}\,dx$ (b) $\displaystyle\int \frac{2x^2-3x}{(x^2+1)(3x+2)}\,dx$

EXERCISE SET 7.5 $\boxed{C}$ CAS

1–8 Write out the form of the partial fraction decomposition. (Do not find the numerical values of the coefficients.)

1. $\dfrac{3x-1}{(x-3)(x+6)}$
2. $\dfrac{5}{x(x^2-9)}$
3. $\dfrac{7x+1}{x^3-x^2}$
4. $\dfrac{x^2}{(x-4)^3}$
5. $\dfrac{1-x^2}{x^3(x^2+2)^2}$
6. $\dfrac{3x}{(x-1)(x^2+7)}$
7. $\dfrac{x^3-3x}{(x^2-2)^2}$
8. $\dfrac{1-3x^4}{(x-3)(x^2-2)^2}$

9–34 Evaluate the integral.

9. $\displaystyle\int \frac{dx}{x^2-3x-10}$
10. $\displaystyle\int \frac{dx}{x^2-7x-8}$
11. $\displaystyle\int \frac{22x+88}{2x^2+9x-5}\,dx$
12. $\displaystyle\int \frac{5x-5}{3x^2-8x-3}\,dx$
13. $\displaystyle\int \frac{2x^2-9x-9}{x^3-9x}\,dx$
14. $\displaystyle\int \frac{dx}{x^2(x^2-1)}$
15. $\displaystyle\int \frac{x^2-7}{x+3}\,dx$
16. $\displaystyle\int \frac{x^2+1}{x-1}\,dx$
17. $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$
18. $\displaystyle\int \frac{x^2+1}{x^2-3x+2}\,dx$
19. $\displaystyle\int \frac{2x-3}{x^2-3x-10}\,dx$
20. $\displaystyle\int \frac{3x+1}{3x^2+2x-1}\,dx$
21. $\displaystyle\int \frac{x^5+x^2+2}{x^3-x}\,dx$
22. $\displaystyle\int \frac{x^5-4x^3+1}{x^3-4x}\,dx$
23. $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$
24. $\displaystyle\int \frac{3x^2-x+1}{x^3-x^2}\,dx$

✓ QUICK CHECK EXERCISES 7.4

(See page 514 for answers.)

1. For each expression, give a trigonometric substitution that will eliminate the radical.
   (a) $\sqrt{a^2 - x^2}$ ______ (b) $\sqrt{a^2 + x^2}$ ______
   (c) $\sqrt{x^2 - a^2}$ ______
2. If $x = 2\sec\theta$ and $0 < \theta < \pi/2$, then
   (a) $\sin\theta =$ ______ (b) $\cos\theta =$ ______
   (c) $\tan\theta =$ ______.
3. In each part, state the trigonometric substitution that you would try first to evaluate the integral. Do not evaluate the integral.
   (a) $\int \sqrt{9 + x^2}\, dx$ ______
   (b) $\int \sqrt{9 - x^2}\, dx$ ______
   (c) $\int \sqrt{1 - 9x^2}\, dx$ ______
   (d) $\int \sqrt{x^2 - 9}\, dx$ ______
   (e) $\int \sqrt{9 + 3x^2}\, dx$ ______
   (f) $\int \sqrt{1 + (9x)^2}\, dx$ ______
4. In each part, determine the substitution u.
   (a) $\displaystyle\int \frac{1}{x^2 - 2x + 10}\, dx = \int \frac{1}{u^2 + 3^2}\, du$;
   $u =$ ______
   (b) $\int \sqrt{x^2 - 6x + 8}\, dx = \int \sqrt{u^2 - 1}\, du$;
   $u =$ ______
   (c) $\int \sqrt{12 - 4x - x^2}\, dx = \int \sqrt{4^2 - u^2}\, du$;
   $u =$ ______

EXERCISE SET 7.4 ☐C CAS

1–26 Evaluate the integral.

1. $\int \sqrt{9 - x^2}\, dx$
2. $\int \sqrt{1 - 4x^2}\, dx$
3. $\displaystyle\int \frac{x^2}{\sqrt{4 - x^2}}\, dx$
4. $\displaystyle\int \frac{dx}{x^2\sqrt{9 - x^2}}$
5. $\displaystyle\int \frac{dx}{(4 + x^2)^2}$
6. $\displaystyle\int \frac{x^2}{\sqrt{7 + x^2}}\, dx$
7. $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x}\, dx$
8. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 - 4}}$
9. $\displaystyle\int \frac{3x^3}{\sqrt{1 - x^2}}\, dx$
10. $\int x^3\sqrt{5 - x^2}\, dx$
11. $\displaystyle\int \frac{dx}{x^2\sqrt{9x^2 - 4}}$
12. $\displaystyle\int \frac{\sqrt{1 + t^2}}{t}\, dt$
13. $\displaystyle\int \frac{dx}{(1 - x^2)^{3/2}}$
14. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 + 25}}$
15. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 16}}$
16. $\displaystyle\int \frac{dx}{1 + 2x^2 + x^4}$
17. $\displaystyle\int \frac{dx}{(4x^2 - 9)^{3/2}}$
18. $\displaystyle\int \frac{3x^3}{\sqrt{x^2 - 25}}\, dx$
19. $\int e^x\sqrt{1 - e^{2x}}\, dx$
20. $\displaystyle\int \frac{\cos\theta}{\sqrt{2 - \sin^2\theta}}\, d\theta$
21. $\displaystyle\int_0^1 5x^3\sqrt{1 - x^2}\, dx$
22. $\displaystyle\int_0^{1/2} \frac{dx}{(1 - x^2)^2}$
23. $\displaystyle\int_{\sqrt{2}}^2 \frac{dx}{x^2\sqrt{x^2 - 1}}$
24. $\displaystyle\int_{\sqrt{2}}^2 \frac{\sqrt{2x^2 - 4}}{x}\, dx$
25. $\displaystyle\int_1^3 \frac{dx}{x^4\sqrt{x^2 + 3}}$
26. $\displaystyle\int_0^3 \frac{x^3}{(3 + x^2)^{5/2}}\, dx$

27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. An integrand involving a radical of the form $\sqrt{x^2 - a^2}$ suggests the substitution $x = a\cos\theta$.
28. The trigonometric substitution $x = a\sin\theta$ is made with the restriction $0 \le \theta \le \pi$.
29. An integrand involving a radical of the form $\sqrt{a^2 - x^2}$ suggests the substitution $x = a\sin\theta$.
30. The area enclosed by the ellipse $x^2 + 16y^2 = 1$ is 2π.

FOCUS ON CONCEPTS

31. The integral
$$\int \frac{x}{x^2 + 9}\, dx$$
can be evaluated either by a trigonometric substitution or by the substitution $u = x^2 + 9$. Do it both ways and show that the results are equivalent.

32. The integral
$$\int \frac{x^2}{x^2 + 4}\, dx$$
can be evaluated either by a trigonometric substitution or by algebraically rewriting the numerator of the integrand as $(x^2 + 4) - 4$. Do it both ways and show that the results are equivalent.

33. Find the arc length of the curve $y = \ln x$ from $x = 1$ to $x = 3$.
34. Find the arc length of the curve $y = x^2$ from $x = 0$ to $x = 1$.

11. $\int \sin^2 x \cos^2 x\, dx$

12. $\int \sin^2 x \cos^4 x\, dx$

13. $\int \sin 3x \cos 4x\, dx$

14. $\int \sin 3\theta \cos 2\theta\, d\theta$

15. $\int \sin x \cos(x/2)\, dx$

16. $\int \cos^{1/3} x \sin x\, dx$

17. $\int_0^{\pi/2} \cos^3 x\, dx$

18. $\int_0^{\pi/2} \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\, dx$

19. $\int_0^{\pi/3} \sin^6 3x \cos^3 3x\, dx$

20. $\int_{-\pi}^{\pi} \cos^2 4\theta\, d\theta$

21. $\int_0^{\pi/3} \sin 4x \cos 2x\, dx$

22. $\int_0^{2\pi} \sin^2 kx\, dx$

23. $\int \sec^2(2x-1)\, dx$

24. $\int \tan 6x\, dx$

25. $\int e^{-x} \tan(e^{-x})\, dx$

26. $\int \cot 4x\, dx$

27. $\int \sec 7x\, dx$

28. $\int \frac{\sec(\sqrt{x})}{\sqrt{x}}\, dx$

29. $\int \tan^2 x \sec^2 x\, dx$

30. $\int \tan^3 x \sec^4 x\, dx$

31. $\int \tan 4x \sec^4 4x\, dx$

32. $\int \tan^4 \theta \sec^4 \theta\, d\theta$

33. $\int \sec^3 x \tan^3 x\, dx$

34. $\int \tan^5 \theta \sec \theta\, d\theta$

35. $\int \tan^4 x \sec x\, dx$

36. $\int \tan^2 x \sec^3 x\, dx$

37. $\int \tan t \sec^3 t\, dt$

38. $\int \tan x \sec^5 x\, dx$

39. $\int \sec^4 x\, dx$

40. $\int \sec^5 x\, dx$

41. $\int \tan^3 4x\, dx$

42. $\int \tan^4 x\, dx$

43. $\int \sqrt{\tan x} \sec^4 x\, dx$

44. $\int \tan x \sec^{3/2} x\, dx$

45. $\int_0^{\pi/8} \tan^2 2x\, dx$

46. $\int_0^{\pi/6} \sec^3 2\theta \tan 2\theta\, d\theta$

47. $\int_0^{\pi/2} \tan^5 \frac{x}{2}\, dx$

48. $\int_0^{1/4} \sec \pi x \tan \pi x\, dx$

49. $\int \cot^3 x \csc^3 x\, dx$

50. $\int \cot^2 3t \sec 3t\, dt$

51. $\int \cot^3 x\, dx$

52. $\int \csc^4 x\, dx$

53–56 True–False Determine whether the statement is true or false. Explain your answer.

53. The trigonometric identity
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$
is often useful for evaluating integrals of the form $\int \sin^m x \cos^n x\, dx$.

54. To evaluate $\int \sin^8 x \cos^5 x\, dx$, use the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.

55. To evaluate $\int \sin^5 x \cos^8 x\, dx$, use the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.

56. The integral $\int \tan^4 x \sec^5 x\, dx$ is equivalent to one whose integrand is a polynomial in $\sec x$.

57. Let m, n be distinct nonnegative integers. Use Formulas (16)–(18) to prove:
(a) $\int_0^{2\pi} \sin mx \cos nx\, dx = 0$
(b) $\int_0^{2\pi} \sin mx \sin nx\, dx = 0.$
(c) $\int_0^{2\pi} \cos mx \cos nx\, dx = 0$

58. Evaluate the integrals in Exercise 57 when m and n denote the *same* nonnegative integer.

59. Find the arc length of the curve $y = \ln(\cos x)$ over the interval $[0, \pi/4]$.

60. Find the volume of the solid generated when the region enclosed by $y = \tan x$, $y = 1$, and $x = 0$ is revolved about the x-axis.

61. Find the volume of the solid that results when the region enclosed by $y = \cos x$, $y = \sin x$, $x = \pi/4$, and $x = \pi/2$ is revolved about the x-axis.

62. The region bounded below by the x-axis and above by the portion of $y = \sin x$ from $x = 0$ to $x = \pi$ is revolved about the x-axis. Find the volume of the resulting solid.

63. Use Formula (27) to show that if the length of the equatorial line on a Mercator projection is L, then the vertical distance D between the latitude lines at $\alpha°$ and $\beta°$ on the same side of the equator (where $\alpha < \beta$) is
$$D = \frac{L}{2\pi}\ln\left|\frac{\sec\beta° + \tan\beta°}{\sec\alpha° + \tan\alpha°}\right|$$

64. Suppose that the equator has a length of 100 cm on a Mercator projection. In each part, use the result in Exercise 63 to answer the question.
(a) What is the vertical distance on the map between the equator and the line at 20° north latitude?
(b) What is the vertical distance on the map between New Orleans, Louisiana, at 30° north latitude and Winnipeg, Canada, at 50° north latitude?

FOCUS ON CONCEPTS

65. (a) Show that
$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$
(b) Show that the result in part (a) can also be written as
$$\int \csc x\, dx = \ln|\csc x - \cot x| + C$$
and
$$\int \csc x\, dx = \ln\left|\tan \tfrac{1}{2}x\right| + C$$

✔ QUICK CHECK EXERCISES 7.2

(See page 500 for answers.)

1. (a) If $G'(x) = g(x)$, then
$$\int f(x)g(x)\,dx = f(x)G(x) - \underline{\qquad}$$
(b) If $u = f(x)$ and $v = G(x)$, then the formula in part (a) can be written in the form $\int u\,dv = \underline{\qquad}$.

2. Find an appropriate choice of u and dv for integration by parts of each integral. Do not evaluate the integral.
(a) $\int x \ln x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(b) $\int (x-2)\sin x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(c) $\int \sin^{-1} x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(d) $\int \dfrac{x}{\sqrt{x-1}}\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$

3. Use integration by parts to evaluate the integral.
(a) $\int xe^{2x}\,dx$ (b) $\int \ln(x-1)\,dx$
(c) $\int_0^{\pi/6} x \sin 3x\,dx$

4. Use a reduction formula to evaluate $\int \sin^3 x\,dx$.

EXERCISE SET 7.2

1–38 Evaluate the integral.

1. $\int xe^{-2x}\,dx$
2. $\int xe^{4x}\,dx$
3. $\int x^2e^{x}\,dx$
4. $\int x^2e^{-2x}\,dx$
5. $\int x^2 \cos x\,dx$
6. $\int x \cos 2x\,dx$
7. $\int x \sin 3x\,dx$
8. $\int x^2 \sin x\,dx$
9. $\int x \ln x\,dx$
10. $\int \sqrt{x} \ln x\,dx$
11. $\int (\ln x)^2\,dx$
12. $\int \dfrac{\ln x}{\sqrt{x}}\,dx$
13. $\int \ln(3x-2)\,dx$
14. $\int \ln(x^2+16)\,dx$
15. $\int \sin^{-1} 2x\,dx$
16. $\int \cos^{-1}(2x)\,dx$
17. $\int \tan^{-1}(3x)\,dx$
18. $\int x \tan^{-1} x\,dx$
19. $\int \sin(\ln x)\,dx$
20. $\int \cos(\ln x)\,dx$
21. $\int e^x \sin x\,dx$
22. $\int e^{3x} \cos 2x\,dx$
23. $\int x \sec^2 x\,dx$
24. $\int x \tan^2 x\,dx$
25. $\int x^3 e^{x^2}\,dx$
26. $\int \dfrac{xe^x}{(x+1)^2}\,dx$
27. $\int_0^2 xe^{3x}\,dx$
28. $\int_0^1 xe^{-3x}\,dx$
29. $\int_1^e x^2 \ln x\,dx$
30. $\int_{\sqrt{e}}^{e} \dfrac{\ln x}{x^2}\,dx$
31. $\int_{-1}^{1} \ln(x+2)\,dx$
32. $\int_0^{\sqrt{3}/2} \sin^{-1} x\,dx$
33. $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$
34. $\int_1^2 x \sec^{-1} x\,dx$
35. $\int_0^{\pi} x \sin 2x\,dx$
36. $\int_0^{\pi} (x + x\cos x)\,dx$
37. $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$
38. $\int_0^2 \ln(x^2+1)\,dx$

39–42 True–False Determine whether the statement is true or false. Explain your answer.

39. The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original.

40. Applying the LIATE strategy to evaluate $\int x^3 \ln x\,dx$, we should choose $u = x^3$ and $dv = \ln x\,dx$.

41. To evaluate $\int \sin(\ln x)\,dx$ using integration by parts, choose $dv = \ln dx$.

42. Tabular integration by parts is useful for integrals of the form $\int p(x)f(x)\,dx$, where $p(x)$ is a polynomial and $f(x)$ can be repeatedly integrated.

43–44 Evaluate the integral by making a u-substitution and then integrating by parts.

43. $\int \sin\sqrt{x}\,dx$
44. $\int e^{\sqrt{x}}\,dx$

45. Prove that tabular integration by parts gives the correct answer for
$$\int p(x)f(x)\,dx$$
where $p(x)$ is any *quadratic polynomial* and $f(x)$ is any function that can be repeatedly integrated.

46. The computations of any integral evaluated by repeated integration by parts can be organized using tabular integration by parts. Use this organization to evaluate $\int e^x \cos x\,dx$ in

8. Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$
   (a) using integration by parts
   (b) using the substitution $u = \sqrt{x^2+1}$.

9–12 Use integration by parts to evaluate the integral. ■

9. $\int xe^{-x}\,dx$ 10. $\int x \sin 3x\,dx$

11. $\int \ln(2x+7)\,dx$ 12. $\int_0^{1/2} \tan^{-1}(2x)\,dx$

13. Evaluate $\int 8x^4 \cos 2x\,dx$ using tabular integration by parts.

14. A particle moving along the x-axis has velocity function $v(t) = t^2e^{-t}$. How far does the particle travel from time $t = 0$ to $t = 4$?

15–20 Evaluate the integral. ■

15. $\int \sin^2 6\theta\,d\theta$ 16. $\int \sin^3 2x \cos^2 2x\,dx$

17. $\int \sin x \cos 3x\,dx$ 18. $\int_0^{\pi/6} \sin 2x \cos 4x\,dx$

19. $\int \sin^4 2x\,dx$ 20. $\int x\cos^5(x^2)\,dx$

21–26 Evaluate the integral by making an appropriate trigonometric substitution. ■

21. $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ 22. $\int \frac{dx}{x^2\sqrt{4-x^2}}$

23. $\int \frac{dx}{\sqrt{x^2-1}}$ 24. $\int \frac{x^2}{\sqrt{x^2-16}}\,dx$

25. $\int \frac{x^2}{\sqrt{9+x^2}}\,dx$ 26. $\int \frac{\sqrt{1+4x^2}}{x}\,dx$

27–32 Evaluate the integral using the method of partial fractions. ■

27. $\int \frac{dx}{x^2+4x-5}$ 28. $\int \frac{dx}{x^2+7x+6}$

29. $\int \frac{x^2+2}{x+2}\,dx$ 30. $\int \frac{x^2+x-16}{(x-1)(x-3)^2}\,dx$

31. $\int \frac{x^2}{(x+2)^3}\,dx$ 32. $\int \frac{dx}{x^3+x}$

33. Consider the integral $\int \frac{1}{x^3-x}\,dx$.
   (a) Evaluate the integral using the substitution $x = \sec\theta$. For what values of x is your result valid?
   (b) Evaluate the integral using the substitution $x = \sin\theta$. For what values of x is your result valid?
   (c) Evaluate the integral using the method of partial fractions. For what values of x is your result valid?

34. Find the area of the region that is enclosed by the curves $y = (x-3)/(x^3+x^2)$, $y = 0$, $x = 1$, and $x = 3$.

35–40 Use the Integral Table to evaluate the integral. ■

35. $\int \sin 7x \cos 10x\,dx$ 36. $\int (x^3-x^2)e^{-x}\,dx$

37. $\int x\sqrt{x-x^2}\,dx$ 38. $\int \frac{dx}{x\sqrt{4x+3}}$

39. $\int \tan^2 2x\,dx$ 40. $\int \frac{3x-1}{2+x^2}\,dx$

41–42 Approximate the integral using (a) the midpoint approximation M_{10}, (b) the trapezoidal approximation T_{10}, and (c) Simpson's rule approximation S_{20}. In each case, find the exact value of the integral and approximate the absolute error. Express your answers to at least four decimal places. ■

41. $\int_1^3 \frac{1}{\sqrt{x+1}}\,dx$ 42. $\int_{-2}^2 \frac{1}{1+x^2}\,dx$

43–44 Use inequalities (12), (13), and (14) of Section 7.7 to find upper bounds on the errors in parts (a), (b), or (c) of the indicated exercise. ■

43. Exercise 41 44. Exercise 42

45–46 Use inequalities (12), (13), and (14) of Section 7.7 to find a number n of subintervals for (a) the midpoint approximation M_n, (b) the trapezoidal approximation T_n, and (c) Simpson's rule approximation S_n to ensure the absolute error will be less than 10^{-4}. ■

45. Exercise 41 46. Exercise 42

47–50 Evaluate the integral if it converges. ■

47. $\int_0^{+\infty} e^{-x}\,dx$ 48. $\int_{-\infty}^2 \frac{dx}{x^2+4}$

49. $\int_0^9 \frac{dx}{\sqrt{9-x}}$ 50. $\int_0^1 \frac{1}{2x-1}\,dx$

51. Find the area that is enclosed between the x-axis and the curve $y = (\ln x - 1)/x^2$ for $x \geq e$.

52. Find the volume of the solid that is generated when the region between the x-axis and the curve $y = e^{-x}$ for $x \geq 0$ is revolved about the y-axis.

53. Find a positive value of a that satisfies the equation

$$\int_0^{+\infty} \frac{1}{x^2+a^2}\,dx = 1$$

54. Consider the following methods for evaluating integrals: u-substitution, integration by parts, partial fractions, reduction formulas, and trigonometric substitutions. In each part, state the approach that you would try first to evaluate the integral. If none of them seems appropriate, then say so. You need not evaluate the integral.
   (a) $\int x \sin x\,dx$ (b) $\int \cos x \sin x\,dx$

(cont.)

(c) $\int \tan^7 x\, dx$ (d) $\int \tan^7 x \sec^2 x\, dx$

(e) $\int \frac{3x^2}{x^3+1}\, dx$ (f) $\int \frac{3x^2}{(x+1)^3}\, dx$

(g) $\int \tan^{-1} x\, dx$ (h) $\int \sqrt{4-x^2}\, dx$

(i) $\int x\sqrt{4-x^2}\, dx$

55–74 Evaluate the integral.

55. $\int \frac{dx}{(3+x^2)^{3/2}}$

56. $\int x \cos 4x\, dx$

57. $\int_0^{\pi/4} \tan^7 \theta\, d\theta$

58. $\int \frac{\cos\theta}{\sin^2\theta - 6\sin\theta + 12}\, d\theta$

59. $\int \sin^2 2x \cos^3 2x\, dx$

60. $\int_0^4 \frac{1}{(x-2)^2}\, dx$

61. $\int e^{2x} \cos 3x\, dx$

62. $\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx$

63. $\int \frac{dx}{(x-1)(x+2)(x-3)}$

64. $\int_0^{1/3} \frac{dx}{(4-9x^2)^2}$

65. $\int_4^8 \frac{\sqrt{x-4}}{x}\, dx$

66. $\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

67. $\int \frac{1}{\sqrt{e^x+1}}\, dx$

68. $\int \frac{dx}{x(x^2+x+1)}$

69. $\int_0^{1/2} \sin^{-1} x\, dx$

70. $\int \tan^5 4x \sec^4 4x\, dx$

71. $\int \frac{x+3}{\sqrt{x^2+2x+2}}\, dx$

72. $\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta}\, d\theta$

73. $\int_a^{+\infty} \frac{x}{(x^2+1)^2}\, dx$

74. $\int_0^{+\infty} \frac{dx}{a^2+b^2x^2}, \quad a, b > 0$

CHAPTER 7 MAKING CONNECTIONS

C CAS

1. Recall from Theorem 6.3.1 and the discussion preceding it that if $f'(x) > 0$, then the function f is increasing and has an inverse function. Parts (a), (b), and (c) of this problem show that if this condition is satisfied and if f' is continuous, then a definite integral of f^{-1} can be expressed in terms of a definite integral of f.

(a) Use integration by parts to show that

$$\int_a^b f(x)\, dx = bf(b) - af(a) - \int_a^b xf'(x)\, dx$$

(b) Use the result in part (a) to show that if $y = f(x)$, then

$$\int_a^b f(x)\, dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\, dy$$

(c) Show that if we let $\alpha = f(a)$ and $\beta = f(b)$, then the result in part (b) can be written as

$$\int_\alpha^\beta f^{-1}(x)\, dx = \beta f^{-1}(\beta) - \alpha f^{-1}(\alpha) - \int_{f^{-1}(\alpha)}^{f^{-1}(\beta)} f(x)\, dx$$

2. In each part, use the result in Exercise 1 to obtain the equation, and then confirm that the equation is correct by performing the integrations.

(a) $\int_0^{1/2} \sin^{-1} x\, dx = \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \int_0^{\pi/6} \sin x\, dx$

(b) $\int_e^{e^2} \ln x\, dx = (2e^2 - e) - \int_1^2 e^x\, dx$

3. The ***Gamma function***, $\Gamma(x)$, is defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\, dt$$

It can be shown that this improper integral converges if and only if $x > 0$.

(a) Find $\Gamma(1)$.

(b) Prove: $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. [*Hint:* Use integration by parts.]

(c) Use the results in parts (a) and (b) to find $\Gamma(2)$, $\Gamma(3)$, and $\Gamma(4)$; and then make a conjecture about $\Gamma(n)$ for positive integer values of n.

(d) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [*Hint:* See Exercise 64 of Section 7.8.]

(e) Use the results obtained in parts (b) and (d) to show that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$.

4. Refer to the Gamma function defined in Exercise 3 to show that

(a) $\int_0^1 (\ln x)^n\, dx = (-1)^n\Gamma(n+1), \quad n > 0$

[*Hint:* Let $t = -\ln x$.]

(b) $\int_0^{+\infty} e^{-x^n}\, dx = \Gamma\left(\frac{n+1}{n}\right), \quad n > 0.$

[*Hint:* Let $t = x^n$. Use the result in Exercise 3(b).]

C **5.** A ***simple pendulum*** consists of a mass that swings in a vertical plane at the end of a massless rod of length L, as shown in the accompanying figure. Suppose that a simple pendulum is displaced through an angle θ_0 and released from rest. It can be