MA1123 Assignment8
[due Monday 6th December 2015]

1. $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}$
2. $\lim_{x\to \infty}(\ln x)^{\frac{1}{x}}$
3. Page 448 Numbers 10,32
4. Page 469/70 Numbers 28,34,40,44
5. Page 481 Numbers 22,40,45,48
6. Page 484/5/6 Numbers 12,35,50,78.

EXERCISE SET 6.5

Graphing Utility C CAS

1–2 Evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule.

1. (a) $\lim_{x\to 2} \frac{x^2-4}{x^2+2x-8}$ (b) $\lim_{x\to +\infty} \frac{2x-5}{3x+7}$

2. (a) $\lim_{x\to 0} \frac{\sin x}{\tan x}$ (b) $\lim_{x\to 1} \frac{x^2-1}{x^3-1}$

3–6 True–False Determine whether the statement is true or false. Explain your answer.

3. L'Hôpital's rule does not apply to $\lim_{x\to -\infty} \frac{\ln x}{x}$.

4. For any polynomial $p(x)$, $\lim_{x\to +\infty} \frac{p(x)}{e^x} = 0$.

5. If n is chosen sufficiently large, then $\lim_{x\to +\infty} \frac{(\ln x)^n}{x} = +\infty$.

6. $\lim_{x\to 0^+} (\sin x)^{1/x} = 0$

7–43 Find the limits.

7. $\lim_{x\to 0} \frac{e^x-1}{\sin x}$

8. $\lim_{x\to 0} \frac{\sin 2x}{\sin 5x}$

9. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$

10. $\lim_{t\to 0} \frac{te^t}{1-e^t}$

11. $\lim_{x\to \pi^+} \frac{\sin x}{x-\pi}$

12. $\lim_{x\to 0^+} \frac{\sin x}{x^2}$

13. $\lim_{x\to +\infty} \frac{\ln x}{x}$

14. $\lim_{x\to +\infty} \frac{e^{3x}}{x^2}$

15. $\lim_{x\to 0^+} \frac{\cot x}{\ln x}$

16. $\lim_{x\to 0^+} \frac{1-\ln x}{e^{1/x}}$

17. $\lim_{x\to +\infty} \frac{x^{100}}{e^x}$

18. $\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$

19. $\lim_{x\to +\infty} xe^{-x}$

20. $\lim_{x\to \pi^-} (x-\pi)\tan\frac{1}{2}x$

21. $\lim_{x\to +\infty} x\sin\frac{\pi}{x}$

22. $\lim_{x\to 0^+} \tan x \ln x$

23. $\lim_{x\to \pi/2^-} \sec 3x \cos 5x$

24. $\lim_{x\to \pi} (x-\pi)\cot x$

25. $\lim_{x\to +\infty} (1-3/x)^x$

26. $\lim_{x\to 0} (1+2x)^{-3/x}$

27. $\lim_{x\to 0} (e^x+x)^{1/x}$

28. $\lim_{x\to +\infty} (1+a/x)^{bx}$

29. $\lim_{x\to 1} (2-x)^{\tan[(\pi/2)x]}$

30. $\lim_{x\to +\infty} [\cos(2/x)]^{x^2}$

31. $\lim_{x\to 0} (\csc x - 1/x)$

32. $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{\cos 3x}{x^2}\right)$

33. $\lim_{x\to +\infty} (\sqrt{x^2+x} - x)$

34. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

35. $\lim_{x\to +\infty} [x - \ln(x^2+1)]$

36. $\lim_{x\to +\infty} [\ln x - \ln(1+x)]$

37. $\lim_{x\to 0^+} x^{\sin x}$

38. $\lim_{x\to 0^+} (e^{2x}-1)^x$

39. $\lim_{x\to 0^+} \left[-\frac{1}{\ln x}\right]^x$

40. $\lim_{x\to +\infty} x^{1/x}$

41. $\lim_{x\to +\infty} (\ln x)^{1/x}$

42. $\lim_{x\to 0^+} (-\ln x)^x$

43. $\lim_{x\to \pi/2^-} (\tan x)^{(\pi/2)-x}$

44. Show that for any positive integer n

(a) $\lim_{x\to +\infty} \frac{\ln x}{x^n} = 0$ (b) $\lim_{x\to +\infty} \frac{x^n}{\ln x} = +\infty$.

FOCUS ON CONCEPTS

45. (a) Find the error in the following calculation:

$$\lim_{x\to 1} \frac{x^3-x^2+x-1}{x^3-x^2} = \lim_{x\to 1} \frac{3x^2-2x+1}{3x^2-2x} = \lim_{x\to 1} \frac{6x-2}{6x-2} = 1$$

(b) Find the correct limit.

46. (a) Find the error in the following calculation:

$$\lim_{x\to 2} \frac{e^{3x^2-12x+12}}{x^4-16} = \lim_{x\to 2} \frac{(6x-12)e^{3x^2-12x+12}}{4x^3} = 0$$

(b) Find the correct limit.

47–50 Make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule.

47. $\lim_{x\to +\infty} \frac{\ln(\ln x)}{\sqrt{x}}$

48. $\lim_{x\to 0^+} x^x$

49. $\lim_{x\to 0^+} (\sin x)^{3/\ln x}$

50. $\lim_{x\to (\pi/2)^-} \frac{4\tan x}{1+\sec x}$

51–54 Make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule.

51. $y = \ln x - e^x$

52. $y = x - \ln(1+2e^x)$

53. $y = (\ln x)^{1/x}$

54. $y = \left(\frac{x+1}{x+2}\right)^x$

55. Limits of the type

$$0/\infty,\quad \infty/0,\quad 0^\infty,\quad \infty\cdot\infty,\quad +\infty+(+\infty),$$
$$+\infty-(-\infty),\quad -\infty+(-\infty),\quad -\infty-(+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\lim_{x\to 0^+} \frac{x}{\ln x}$ (b) $\lim_{x\to +\infty} \frac{x^3}{e^{-x}}$

(c) $\lim_{x\to (\pi/2)^-} (\cos x)^{\tan x}$ (d) $\lim_{x\to 0^+} (\ln x)\cot x$

(e) $\lim_{x\to 0^+} \left(\frac{1}{x} - \ln x\right)$ (f) $\lim_{x\to -\infty} (x+x^3)$

EXERCISE SET 6.7 Graphing Utility C CAS

1. Given that $\theta = \tan^{-1}\left(\frac{4}{3}\right)$, find the exact values of $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

2. Given that $\theta = \sec^{-1} 2.6$, find the exact values of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, and $\csc\theta$.

3. For which values of x is it true that
(a) $\cos^{-1}(\cos x) = x$ (b) $\cos(\cos^{-1} x) = x$
(c) $\tan^{-1}(\tan x) = x$ (d) $\tan(\tan^{-1} x) = x$?

4–5 Find the exact value of the given quantity.

4. $\sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right]$ 5. $\sin\left[2\cos^{-1}\left(\frac{3}{5}\right)\right]$

6–7 Complete the identities using the triangle method (Figure 6.7.3).

6. (a) $\sin(\cos^{-1} x) = ?$ (b) $\tan(\cos^{-1} x) = ?$
(c) $\csc(\tan^{-1} x) = ?$ (d) $\sin(\tan^{-1} x) = ?$

7. (a) $\cos(\tan^{-1} x) = ?$ (b) $\tan(\cos^{-1} x) = ?$
(c) $\sin(\sec^{-1} x) = ?$ (d) $\cot(\sec^{-1} x) = ?$

8. (a) Use a calculating utility set to radian measure to make tables of values of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for $x = -1, -0.8, -0.6, \ldots, 0, 0.2, \ldots, 1$. Round your answers to two decimal places.
(b) Plot the points obtained in part (a), and use the points to sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$. Confirm that your sketches agree with those in Figure 6.7.1.
(c) Use your graphing utility to graph $y = \sin^{-1} x$ and $y = \cos^{-1} x$; confirm that the graphs agree with those in Figure 6.7.1.

9. In each part, sketch the graph and check your work with a graphing utility.
(a) $y = \sin^{-1} 2x$ (b) $y = \tan^{-1} \frac{1}{2}x$

10. The *law of cosines* states that
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$
where a, b, and c are the lengths of the sides of a triangle and θ is the angle formed by sides a and b. Find θ, to the nearest degree, for the triangle with $a = 2$, $b = 3$, and $c = 4$.

11–12 Use a calculating utility to approximate the solution of each equation. Where radians are used, express your answer to four decimal places, and where degrees are used, express it to the nearest tenth of a degree. [*Note:* In each part, the solution is not in the range of the relevant inverse trigonometric function.]

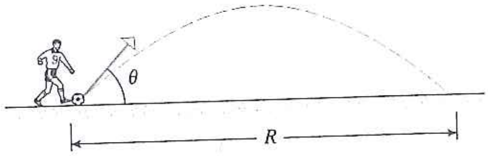
11. (a) $\sin x = 0.37$, $\pi/2 < x < \pi$
(b) $\sin\theta = -0.61$, $180° < \theta < 270°$

12. (a) $\cos x = -0.85$, $\pi < x < 3\pi/2$
(b) $\cos\theta = 0.23$, $-90° < \theta < 0°$

FOCUS ON CONCEPTS

13. (a) Use a calculating utility to evaluate the expressions $\sin^{-1}(\sin^{-1} 0.25)$ and $\sin^{-1}(\sin^{-1} 0.9)$, and explain what you think is happening in the second calculation.
(b) For what values of x in the interval $-1 \le x \le 1$ will your calculating utility produce a real value for the function $\sin^{-1}(\sin^{-1} x)$?

14. A soccer player kicks a ball with an initial speed of 14 m/s at an angle θ with the horizontal (see the accompanying figure). The ball lands 18 m down the field. If air resistance is neglected, then the ball will have a parabolic trajectory and the horizontal range R will be given by
$$R = \frac{v^2}{g}\sin 2\theta$$
where v is the initial speed of the ball and g is the acceleration due to gravity. Using $g = 9.8\ \text{m/s}^2$, approximate two values of θ, to the nearest degree, at which the ball could have been kicked. Which angle results in the shorter time of flight? Why?



▲ Figure Ex-14

15–26 Find dy/dx.

15. $y = \sin^{-1}(3x)$ 16. $y = \cos^{-1}\left(\dfrac{x+1}{2}\right)$

17. $y = \sin^{-1}(1/x)$ 18. $y = \cos^{-1}(\cos x)$

19. $y = \tan^{-1}(x^3)$ 20. $y = \sec^{-1}(x^5)$

21. $y = (\tan x)^{-1}$ 22. $y = \dfrac{1}{\tan^{-1} x}$

23. $y = e^x \sec^{-1} x$ 24. $y = \ln(\cos^{-1} x)$

25. $y = \sin^{-1} x + \cos^{-1} x$ 26. $y = x^2(\sin^{-1} x)^3$

27–28 Find dy/dx by implicit differentiation.

27. $x^3 + x\tan^{-1} y = e^y$ 28. $\sin^{-1}(xy) = \cos^{-1}(x - y)$

29–30 Evaluate the integral and check your answer by differentiating.

29. $\displaystyle\int \left[\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2}\right] dx$

30. $\displaystyle\int \left[\frac{4}{x\sqrt{x^2-1}} + \frac{1+x+x^3}{1+x^2}\right] dx$

31–48 Evaluate the integral.

31. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$ 32. $\displaystyle\int \frac{dx}{1+16x^2}$

33. $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ 34. $\displaystyle\int \frac{t}{t^4+1}\,dt$

35. $\displaystyle\int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$ 36. $\displaystyle\int \frac{\sin\theta}{\cos^2\theta+1}\,d\theta$

37. $\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$ **38.** $\int_{-1}^{1} \frac{dx}{1+x^2}$

39. $\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}$ **40.** $\int_{-\sqrt{2}}^{-2/\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}}$

41. $\int_1^{\sqrt{3}} \frac{\sqrt{\tan^{-1} x}}{1+x^2}\,dx$ **42.** $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-(\ln x)^2}}$

43. $\int_1^3 \frac{dx}{\sqrt{x}(x+1)}$ **44.** $\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}\,dx}{\sqrt{1-e^{-2x}}}$

45. $\int_0^1 \frac{x}{\sqrt{4-3x^4}}\,dx$ **46.** $\int_1^2 \frac{1}{\sqrt{x}\sqrt{4-x}}\,dx$

47. $\int_0^{1/\sqrt{3}} \frac{1}{1+9x^2}\,dx$ **48.** $\int_1^{\sqrt{2}} \frac{x}{3+x^4}\,dx$

49–50 Evaluate the integrals with the aid of Formulas (23), (24), and (25).

49. (a) $\int \frac{dx}{\sqrt{9-x^2}}$ (b) $\int \frac{dx}{5+x^2}$ (c) $\int \frac{dx}{x\sqrt{x^2-\pi}}$

50. (a) $\int \frac{e^x}{4+e^{2x}}\,dx$ (b) $\int \frac{dx}{\sqrt{9-4x^2}}$ (c) $\int \frac{dy}{y\sqrt{5y^2-3}}$

51–54 True–False Determine whether the statement is true or false. Explain your answer.

51. By definition, $\sin^{-1}(\sin x) = x$ for all real numbers x.

52. The range of the inverse tangent function is the interval $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$.

53. The graph of $y = \sec^{-1} x$ has a horizontal asymptote.

54. We can conclude from the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ that $\sin^{-1} x + \cos^{-1} x$ is constant.

FOCUS ON CONCEPTS

55–56 The function $\cot^{-1} x$ is defined to be the inverse of the restricted cotangent function

$$\cot x, \quad 0 < x < \pi$$

and the function $\csc^{-1} x$ is defined to be the inverse of the restricted cosecant function

$$\csc x, \quad -\pi/2 \le x \le \pi/2, \quad x \ne 0$$

Use these definitions in these and in all subsequent exercises that involve these functions.

55. (a) Sketch the graphs of $\cot^{-1} x$ and $\csc^{-1} x$.
(b) Find the domain and range of $\cot^{-1} x$ and $\csc^{-1} x$.

56. Show that

(a) $\cot^{-1} x = \begin{cases} \tan^{-1}(1/x), & \text{if } x > 0 \\ \pi + \tan^{-1}(1/x), & \text{if } x < 0 \end{cases}$

(b) $\sec^{-1} x = \cos^{-1} \frac{1}{x}$, if $|x| \ge 1$

(c) $\csc^{-1} x = \sin^{-1} \frac{1}{x}$, if $|x| \ge 1$.

57. Most scientific calculators have keys for the values of only $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$. The formulas in Exercise 54 show how a calculator can be used to obtain values of $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ for positive values of x. Use these formulas and a calculator to find numerical values for each of the following inverse trigonometric functions. Express your answers in degrees, rounded to the nearest tenth of a degree.
(a) $\cot^{-1} 0.7$ (b) $\sec^{-1} 1.2$ (c) $\csc^{-1} 2.3$

58. A camera is positioned x feet from the base of a missile launching pad (see the accompanying figure). If a missile of length a feet is launched vertically, show that when the base of the missile is b feet above the camera lens, the angle θ subtended at the lens by the missile is

$$\theta = \cot^{-1} \frac{x}{a+b} - \cot^{-1} \frac{x}{b}$$

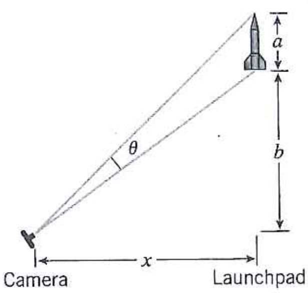


Figure Ex-58

59. Use identity (5) and Formula (14) to obtain the derivative of $y = \cos^{-1} x$.

60. (a) Use Formula (2) in Section 6.3 to prove that

$$\frac{d}{dx}[\cot^{-1} x]\Big|_{x=0} = -1$$

(b) Use part (a) above, part (a) of Exercise 56 and the chain rule to show that

$$\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{1+x^2}$$

for $-\infty < x < +\infty$.

(c) Conclude from part (b) that

$$\frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx}$$

for $-\infty < u < +\infty$.

61. (a) Use part (c) of Exercise 56 and the chain rule to show that

$$\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$

for $1 < |x|$.

(b) Conclude from part (a) that

$$\frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

for $1 < |u|$.

(c) Use Equation (5) and parts (b) and (c) of Exercise 56 to show that if $|x| \ge 1$ then, $\sec^{-1} x + \csc^{-1} x = \pi/2$. Conclude from part (a) that

$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

5. $\int \cosh x\,dx =$ ______ $\int \sinh x\,dx =$ ______
$\int \tanh x\,dx =$ ______

6. $\dfrac{d}{dx}[\cosh^{-1} x] =$ ______ $\dfrac{d}{dx}[\sinh^{-1} x] =$ ______
$\dfrac{d}{dx}[\tanh^{-1} x] =$ ______

EXERCISE SET 6.8 Graphing Utility

1–2 Approximate the expression to four decimal places.

1. (a) $\sinh 3$ (b) $\cosh(-2)$ (c) $\tanh(\ln 4)$
(d) $\sinh^{-1}(-2)$ (e) $\cosh^{-1} 3$ (f) $\tanh^{-1} \frac{3}{4}$

2. (a) $\operatorname{csch}(-1)$ (b) $\operatorname{sech}(\ln 2)$ (c) $\coth 1$
(d) $\operatorname{sech}^{-1} \frac{1}{2}$ (e) $\coth^{-1} 3$ (f) $\operatorname{csch}^{-1}(-\sqrt{3})$

3. Find the exact numerical value of each expression.
(a) $\sinh(\ln 3)$ (b) $\cosh(-\ln 2)$
(c) $\tanh(2\ln 5)$ (d) $\sinh(-3\ln 2)$

4. In each part, rewrite the expression as a ratio of polynomials.
(a) $\cosh(\ln x)$ (b) $\sinh(\ln x)$
(c) $\tanh(2\ln x)$ (d) $\cosh(-\ln x)$

5. In each part, a value for one of the hyperbolic functions is given at an unspecified positive number x_0. Use appropriate identities to find the exact values of the remaining five hyperbolic functions at x_0.
(a) $\sinh x_0 = 2$ (b) $\cosh x_0 = \frac{5}{4}$ (c) $\tanh x_0 = \frac{4}{5}$

6. Obtain the derivative formulas for $\operatorname{csch} x$, $\operatorname{sech} x$, and $\coth x$ from the derivative formulas for $\sinh x$, $\cosh x$, and $\tanh x$.

7. Find the derivatives of $\cosh^{-1} x$ and $\tanh^{-1} x$ by differentiating the formulas in Theorem 6.8.4.

8. Find the derivatives of $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$ by differentiating the equations $x = \sinh y$, $x = \cosh y$, and $x = \tanh y$ implicitly.

9–28 Find dy/dx.

9. $y = \sinh(4x - 8)$
10. $y = \cosh(x^4)$
11. $y = \coth(\ln x)$
12. $y = \ln(\tanh 2x)$
13. $y = \operatorname{csch}(1/x)$
14. $y = \operatorname{sech}(e^{2x})$
15. $y = \sqrt{4x + \cosh^2(5x)}$
16. $y = \sinh^3(2x)$
17. $y = x^3 \tanh^2(\sqrt{x})$
18. $y = \sinh(\cos 3x)$
19. $y = \sinh^{-1}(\frac{1}{3}x)$
20. $y = \sinh^{-1}(1/x)$
21. $y = \ln(\cosh^{-1} x)$
22. $y = \cosh^{-1}(\sinh^{-1} x)$
23. $y = \dfrac{1}{\tanh^{-1} x}$
24. $y = (\coth^{-1} x)^2$
25. $y = \cosh^{-1}(\cosh x)$
26. $y = \sinh^{-1}(\tanh x)$
27. $y = e^x \operatorname{sech}^{-1}\sqrt{x}$
28. $y = (1 + x\operatorname{csch}^{-1} x)^{10}$

29–44 Evaluate the integrals.

29. $\int \sinh^6 x \cosh x\,dx$
30. $\int \cosh(2x - 3)\,dx$
31. $\int \sqrt{\tanh x}\operatorname{sech}^2 x\,dx$
32. $\int \operatorname{csch}^2(3x)\,dx$
33. $\int \tanh 2x\,dx$
34. $\int \coth^2 x \operatorname{csch}^2 x\,dx$
35. $\int_{\ln 2}^{\ln 3} \tanh x \operatorname{sech}^3 x\,dx$
36. $\int_0^{\ln 3} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$
37. $\int \dfrac{dx}{\sqrt{1 + 9x^2}}$
38. $\int \dfrac{dx}{\sqrt{x^2 - 2}} \quad (x > \sqrt{2})$
39. $\int \dfrac{dx}{\sqrt{1 - e^{2x}}} \quad (x < 0)$
40. $\int \dfrac{\sin\theta\,d\theta}{\sqrt{1 + \cos^2\theta}}$
41. $\int \dfrac{dx}{x\sqrt{1 + 4x^2}}$
42. $\int \dfrac{dx}{\sqrt{9x^2 - 25}} \quad (x > 5/3)$
43. $\int_0^{1/2} \dfrac{dx}{1 - x^2}$
44. $\int_0^{\sqrt{3}} \dfrac{dt}{\sqrt{t^2 + 1}}$

45–48 True–False Determine whether the statement is true or false. Explain your answer.

45. The equation $\cosh x = \sinh x$ has no solutions.

46. Exactly two of the hyperbolic functions are bounded.

47. There is exactly one hyperbolic function $f(x)$ such that for all real numbers a, the equation $f(x) = a$ has a unique solution x.

48. The identities in Theorem 6.8.2 may be obtained from the corresponding trigonometric identities by replacing each trigonometric function with its hyperbolic analogue.

49. Find the area enclosed by $y = \sinh 2x$, $y = 0$, and $x = \ln 3$.

50. Find the volume of the solid that is generated when the region enclosed by $y = \operatorname{sech} x$, $y = 0$, $x = 0$, and $x = \ln 2$ is revolved about the x-axis.

51. Find the volume of the solid that is generated when the region enclosed by $y = \cosh 2x$, $y = \sinh 2x$, $x = 0$, and $x = 5$ is revolved about the x-axis.

52. Approximate the positive value of the constant a such that the area enclosed by $y = \cosh ax$, $y = 0$, $x = 0$, and $x = 1$ is 2 square units. Express your answer to at least five decimal places.

53. Find the arc length of the catenary $y = \cosh x$ between $x = 0$ and $x = \ln 2$.

54. Find the arc length of the catenary $y = a\cosh(x/a)$ between $x = 0$ and $x = x_1$ $(x_1 > 0)$.

55. In parts (a)–(f) find the limits, and confirm that they are consistent with the graphs in Figures 6.8.1 and 6.8.6.

CHAPTER 6 REVIEW EXERCISES

☐ Graphing Utility

1. In each part, find $f^{-1}(x)$ if the inverse exists.
   (a) $f(x) = (e^x)^2 + 1$
   (b) $f(x) = \sin\left(\frac{1-2x}{x}\right), \quad \frac{2}{4+\pi} \le x \le \frac{2}{4-\pi}$
   (c) $f(x) = \frac{1}{1 + 3\tan^{-1} x}$

2. (a) State the restrictions on the domains of $\sin x$, $\cos x$, $\tan x$, and $\sec x$ that are imposed to make those functions one-to-one in the definitions of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, and $\sec^{-1} x$.
   (b) Sketch the graphs of the restricted trigonometric functions in part (a) and their inverses.

3. In each part, find the exact numerical value of the given expression.
   (a) $\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)]$
   (b) $\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)]$

4. In each part, sketch the graph, and check your work with a graphing utility.
   (a) $f(x) = 3\sin^{-1}(x/2)$
   (b) $f(x) = \cos^{-1} x - \pi/2$
   (c) $f(x) = 2\tan^{-1}(-3x)$
   (d) $f(x) = \cos^{-1} x + \sin^{-1} x$

5. Suppose that the graph of $y = \log x$ is drawn with equal scales of 1 inch per unit in both the x- and y-directions. If a bug wants to walk along the graph until it reaches a height of 5 ft above the x-axis, how many miles to the right of the origin will it have to travel?

6. Find the largest value of a such that the function $f(x) = xe^{-x}$ has an inverse on the interval $(-\infty, a]$.

7. Express the following function as a rational function of x:

$$3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1)$$

8. Suppose that $y = Ce^{kt}$, where C and k are constants, and let $Y = \ln y$. Show that the graph of Y versus t is a line, and state its slope and Y-intercept.

9. (a) Sketch the curves $y = \pm e^{-x/2}$ and $y = e^{-x/2}\sin 2x$ for $-\pi/2 \le x \le 3\pi/2$ in the same coordinate system, and check your work using a graphing utility.
   (b) Find all x-intercepts of the curve $y = e^{-x/2}\sin 2x$ in the stated interval, and find the x-coordinates of all points where this curve intersects the curves $y = \pm e^{-x/2}$.

10. Suppose that a package of medical supplies is dropped from a helicopter straight down by parachute into a remote area. The velocity v (in feet per second) of the package t seconds after it is released is given by $v = 24.61(1 - e^{-1.3t})$.
   (a) Graph v versus t.
   (b) Show that the graph has a horizontal asymptote $v = c$.
   (c) The constant c is called the ***terminal velocity***. Explain what the terminal velocity means in practical terms.
   (d) Can the package actually reach its terminal velocity? Explain.
   (e) How long does it take for the package to reach 98% of its terminal velocity?

11. A breeding group of 20 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, N, after t years will be given by the formula

$$N = \frac{220}{1 + 10(0.83^t)}$$

and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 80.
   (a) Graph N versus t.
   (b) How many years must the state of Colorado maintain a program to care for the sheep?
   (c) How many bighorn sheep can the environment in the protected area support? [*Hint:* Examine the graph of N versus t for large values of t.]

12. An oven is preheated and then remains at a constant temperature. A potato is placed in the oven to bake. Suppose that the temperature T (in °F) of the potato t minutes later is given by $T = 400 - 325(0.97^t)$. The potato will be considered done when its temperature is anywhere between 260°F and 280°F.
   (a) During what interval of time would the potato be considered done?
   (b) How long does it take for the difference between the potato and oven temperatures to be cut in half?

13. (a) Show that the graphs of $y = \ln x$ and $y = x^{0.2}$ intersect.
   (b) Approximate the solution(s) of the equation $\ln x = x^{0.2}$ to three decimal places.

14. (a) Show that for $x > 0$ and $k \ne 0$ the equations

$$x^k = e^x \quad \text{and} \quad \frac{\ln x}{x} = \frac{1}{k}$$

have the same solutions.
   (b) Use the graph of $y = (\ln x)/x$ to determine the values of k for which the equation $x^k = e^x$ has two distinct positive solutions.
   (c) Estimate the positive solution(s) of $x^8 = e^x$.

15–18 Find the limits.

15. $\lim_{t\to\pi/2^+} e^{\tan t}$

16. $\lim_{\theta\to 0^+} \ln(\sin 2\theta) - \ln(\tan\theta)$

17. $\lim_{x\to+\infty} \left(1 + \frac{3}{x}\right)^{-x}$

18. $\lim_{x\to+\infty} \left(1 + \frac{a}{x}\right)^{bx}, \quad a, b > 0$

19–20 Find dy/dx by first using algebraic properties of the natural logarithm function.

19. $y = \ln\left(\frac{(x+1)(x+2)^2}{(x+3)^3(x+4)^4}\right)$

20. $y = \ln\left(\frac{\sqrt{x}\sqrt[3]{x+1}}{\sin x \sec x}\right)$

21–38 Find dy/dx.

21. $y = \ln 2x$
22. $y = (\ln x)^2$
23. $y = \sqrt[3]{\ln x + 1}$
24. $y = \ln(\sqrt[3]{x+1})$
25. $y = \log(\ln x)$
26. $y = \dfrac{1 + \log x}{1 - \log x}$
27. $y = \ln(x^{3/2}\sqrt{1+x^4})$
28. $y = \ln\left(\dfrac{\sqrt{x}\cos x}{1+x^2}\right)$
29. $y = e^{\ln(x^2+1)}$
30. $y = \ln\left(\dfrac{1+e^x+e^{2x}}{1-e^{3x}}\right)$
31. $y = 2xe^{\sqrt{x}}$
32. $y = \dfrac{a}{1+be^{-x}}$
33. $y = \dfrac{1}{\pi}\tan^{-1} 2x$
34. $y = 2^{\sin^{-1} x}$
35. $y = x^{(e^x)}$
36. $y = (1+x)^{1/x}$
37. $y = \sec^{-1}(2x+1)$
38. $y = \sqrt{\cos^{-1} x^2}$

39–40 Find dy/dx using logarithmic differentiation.

39. $y = \dfrac{x^3}{\sqrt{x^2+1}}$
40. $y = \sqrt[3]{\dfrac{x^2-1}{x^2+1}}$

41. (a) Make a conjecture about the shape of the graph of $y = \frac{1}{2}x - \ln x$, and draw a rough sketch.
(b) Check your conjecture by graphing the equation over the interval $0 < x < 5$ with a graphing utility.
(c) Show that the slopes of the tangent lines to the curve at $x = 1$ and $x = e$ have opposite signs.
(d) What does part (c) imply about the existence of a horizontal tangent line to the curve? Explain.
(e) Find the exact x-coordinates of all horizontal tangent lines to the curve.

42. Recall from Section 6.1 that the loudness β of a sound in decibels (dB) is given by $\beta = 10\log(I/I_0)$, where I is the intensity of the sound in watts per square meter (W/m^2) and I_0 is a constant that is approximately the intensity of a sound at the threshold of human hearing. Find the rate of change of β with respect to I at the point where
(a) $I/I_0 = 10$ (b) $I/I_0 = 100$ (c) $I/I_0 = 1000$.

43. A particle is moving along the curve $y = x\ln x$. Find all values of x at which the rate of change of y with respect to time is three times that of x. [Assume that dx/dt is never zero.]

44. Find the equation of the tangent line to the graph of $y = \ln(5 - x^2)$ at $x = 2$.

45. Find the value of b so that the line $y = x$ is tangent to the graph of $y = \log_b x$. Confirm your result by graphing both $y = x$ and $y = \log_b x$ in the same coordinate system.

46. In each part, find the value of k for which the graphs of $y = f(x)$ and $y = \ln x$ share a common tangent line at their point of intersection. Confirm your result by graphing $y = f(x)$ and $y = \ln x$ in the same coordinate system.
(a) $f(x) = \sqrt{x} + k$ (b) $f(x) = k\sqrt{x}$

47. If f and g are inverse functions and f is differentiable on its domain, must g be differentiable on its domain? Give a reasonable informal argument to support your answer.

48. In each part, find $(f^{-1})'(x)$ using Formula (2) of Section 6.3, and check your answer by differentiating f^{-1} directly.
(a) $f(x) = 3/(x+1)$ (b) $f(x) = \sqrt{e^x}$

49. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

50. Show that the rate of change of $y = 5000e^{1.07x}$ is proportional to y.

51. Show that the function $y = e^{ax}\sin bx$ satisfies

$$y'' - 2ay' + (a^2+b^2)y = 0$$

for any real constants a and b.

52. Show that the function $y = \tan^{-1} x$ satisfies

$$y'' = -2\sin y\cos^3 y$$

53. Suppose that the population of deer on an island is modeled by the equation

$$P(t) = \frac{95}{5 - 4e^{-t/4}}$$

where $P(t)$ is the number of deer t weeks after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

54. The equilibrium constant k of a balanced chemical reaction changes with the absolute temperature T according to the law

$$k = k_0 \exp\left(-\frac{q(T-T_0)}{2T_0T}\right)$$

where k_0, q, and T_0 are constants. Find the rate of change of k with respect to T.

55–56 Find the limit by interpreting the expression as an appropriate derivative.

55. $\displaystyle\lim_{h\to 0}\frac{(1+h)^\pi - 1}{h}$
56. $\displaystyle\lim_{x\to e}\frac{1-\ln x}{(x-e)\ln x}$

57. Suppose that $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. In each of the four possible cases, state whether $\lim[f(x) - g(x)]$ is an indeterminate form, and give a reasonable informal argument to support your answer.

58. (a) Under what conditions will a limit of the form

$$\lim_{x\to a}[f(x)/g(x)]$$

be an indeterminate form?
(b) If $\lim_{x\to a} g(x) = 0$, must $\lim_{x\to a}[f(x)/g(x)]$ be an indeterminate form? Give some examples to support your answer.

59–62 Evaluate the given limit.