Analysis on the Real Line I
Assignment 7

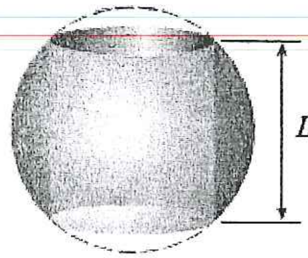
18 November 2013

1. Let V be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ $(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which $V = 3$.

2. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $y = 3$.

3. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

4. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the centre of a sphere. Show that the volume of the remaining solid depends only on the length L of the hole, not on the size of the sphere.



5. Consider the curve $y = x^{2/3}$.

   (a) Sketch the portion of the curve between $x = -1$ and $x = 8$.

   (b) Explain why the following formula cannot be used to find the arc length of the curve sketched in part (a):

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

   (c) Find the arc length of the curve sketched in part (a).

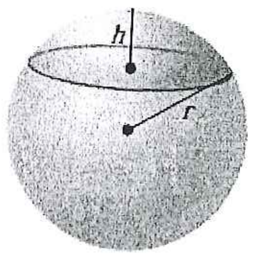
6. The curve segment $y = x^2$ from $x = 1$ to $x = 2$ may also be expressed as the graph of $x = \sqrt{y}$ from $y = 1$ to $y = 4$. Set up two integrals that give the arc length of this curve segment, one by integrating with respect to x, and the other by integrating with respect to y. Demonstrate a substitution that verifies that these two integrals are equal.

7. Let $y = f(x)$ be a smooth curve on the closed interval $[a, b]$. Prove that if m and M are nonnegative numbers such that $m \le |f'(x)| \le M$ for all x in $[a, b]$, then the arc length L of $y = f(x)$ over the interval $[a.b]$ satisfies the inequalities

$$(b-a)\sqrt{1+m^2} \le L \le (b-a)\sqrt{1+M^2}$$

8. Show that the area of the surface of a sphere of radius r is $4\pi r^2$. [*Hint:* Revolve the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis.]

9. The accompanying figure shows a spherical cap of height h cut from a sphere of radius r. Show that the surface area S of the cap is $S = 2\pi rh$. [*Hint*: Revolve an appropriate portion of the circle $x^2 + y^2 = r^2$ about the y-axis.]



10. Let $y = f(x)$ be a smooth curve on the interval $[a, b]$ and assume that $f(x) \ge 0$ for $a \le x \le b$. By the Extreme-Value Theorem, the function f has a maximum value K and a minimum value k on $[a, b]$. Prove: If L is the arc length of the curve $y = f(x)$ between $x = a$ and $x = b$, and if S is the area of the surface that is generated by revolving this curve about the x-axis, then

$$2\pi kL \le S \le 2\pi KL$$

Note (Extreme-Value Theorem):
If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.