MA1123 Assignment6

[due Monday 23rd November 2015]

1. $\int \cos^5 x dx$
2. $\int \sec^3 x \tan x dx$
3. $\int \frac{x}{\sqrt{2x+1}} dx$
4. Page 308 Numbers 29-32,42,45
5. Page 320/321 Numbers 27-30,45(a) (b)
6. Page 335/6 Numbers 15-18,24
7. Page 341 Numbers 44,45,46,47.

21. Find $\int_{-1}^{2} [f(x) + 2g(x)]\, dx$ if

$$\int_{-1}^{2} f(x)\, dx = 5 \quad \text{and} \quad \int_{-1}^{2} g(x)\, dx = -3$$

22. Find $\int_{1}^{4} [3f(x) - g(x)]\, dx$ if

$$\int_{1}^{4} f(x)\, dx = 2 \quad \text{and} \quad \int_{1}^{4} g(x)\, dx = 10$$

23. Find $\int_{1}^{5} f(x)\, dx$ if

$$\int_{0}^{1} f(x)\, dx = -2 \quad \text{and} \quad \int_{0}^{5} f(x)\, dx = 1$$

24. Find $\int_{3}^{-2} f(x)\, dx$ if

$$\int_{-2}^{1} f(x)\, dx = 2 \quad \text{and} \quad \int_{1}^{3} f(x)\, dx = -6$$

25–28 Use Theorem 4.5.4 and appropriate formulas from geometry to evaluate the integrals.

25. $\int_{-1}^{3} (4 - 5x)\, dx$ 26. $\int_{-2}^{2} (1 - 3|x|)\, dx$

27. $\int_{0}^{1} (x + 2\sqrt{1 - x^2})\, dx$ 28. $\int_{-3}^{0} (2 + \sqrt{9 - x^2})\, dx$

29–32 **True–False** Determine whether the statement is true or false. Explain your answer.

29. If $f(x)$ is integrable on $[a, b]$, then $f(x)$ is continuous on $[a, b]$.

30. It is the case that

$$0 < \int_{-1}^{1} \frac{\cos x}{\sqrt{1 + x^2}}\, dx$$

31. If the integral of $f(x)$ over the interval $[a, b]$ is negative, then $f(x) \le 0$ for $a \le x \le b$.

32. The function

$$f(x) = \begin{cases} 0, & x \le 0 \\ x^2, & x > 0 \end{cases}$$

is integrable over every closed interval $[a, b]$.

33–34 Use Theorem 4.5.6 to determine whether the value of the integral is positive or negative.

33. (a) $\int_{2}^{3} \frac{\sqrt{x}}{1 - x}\, dx$ (b) $\int_{0}^{4} \frac{x^2}{3 - \cos x}\, dx$

34. (a) $\int_{-3}^{-1} \frac{x^4}{\sqrt{3 - x}}\, dx$ (b) $\int_{-2}^{2} \frac{x^3 - 9}{|x| + 1}\, dx$

35. Prove that if f is continuous and if $m \le f(x) \le M$ on $[a, b]$, then

$$m(b - a) \le \int_{a}^{b} f(x)\, dx \le M(b - a)$$

36. Find the maximum and minimum values of $\sqrt{x^3 + 2}$ for $0 \le x \le 3$. Use these values, and the inequalities in Exercise 35, to find bounds on the value of the integral

$$\int_{0}^{3} \sqrt{x^3 + 2}\, dx$$

37–38 Evaluate the integrals by completing the square and applying appropriate formulas from geometry.

37. $\int_{0}^{10} \sqrt{10x - x^2}\, dx$ 38. $\int_{0}^{3} \sqrt{6x - x^2}\, dx$

39–40 Evaluate the limit by expressing it as a definite integral over the interval $[a, b]$ and applying appropriate formulas from geometry.

39. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (3x_k^* + 1)\Delta x_k;\ a = 0, b = 1$

40. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{4 - (x_k^*)^2}\, \Delta x_k;\ a = -2, b = 2$

FOCUS ON CONCEPTS

41. Let $f(x) = C$ be a constant function.
(a) Use a formula from geometry to show that

$$\int_{a}^{b} f(x)\, dx = C(b - a)$$

(b) Show that any Riemann sum for $f(x)$ over $[a, b]$ evaluates to $C(b - a)$. Use Definition 4.5.1 to show that

$$\int_{a}^{b} f(x)\, dx = C(b - a)$$

42. Define a function f on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Use Definition 4.5.1 to show that

$$\int_{0}^{1} f(x)\, dx = 1$$

43. It can be shown that every interval contains both rational and irrational numbers. Accepting this to be so, do you believe that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is integrable on a closed interval $[a, b]$? Explain your reasoning.

44. Define the function f by
$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
It follows from Theorem 4.5.8(*b*) that f is not integrable on the interval $[0, 1]$. Prove this to be the case by applying Definition 4.5.1. [*Hint:* Argue that no matter how small the mesh size is for a partition of $[0, 1]$, there will always be a choice of x_1^* that will make the Riemann sum in Definition 4.5.1 as large as we like.]

45. In each part, use Theorems 4.5.2 and 4.5.8 to determine whether the function f is integrable on the interval $[-1, 1]$.

(a) $f(x) = \cos x$

(b) $f(x) = \begin{cases} x/|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(d) $f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

46. **Writing** Write a short paragraph that discusses the similarities and differences between indefinite integrals and definite integrals.

47. **Writing** Write a paragraph that explains informally what it means for a function to be "integrable."

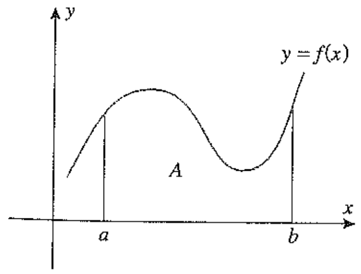
✔ QUICK CHECK ANSWERS 4.5

1. (a) $n = 4$ (b) $2, 3, 4.5, 6.5, 7$ (c) $1, 1.5, 2, 0.5$ (d) 2 2. 3 3. 5 4. (a) -10 (b) 3 (c) 0 (d) -12

4.6 THE FUNDAMENTAL THEOREM OF CALCULUS

In this section we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the "Fundamental Theorem of Calculus." One part of this theorem will relate the rectangle and antiderivative methods for calculating areas, and the second part will provide a powerful method for evaluating definite integrals using antiderivatives.

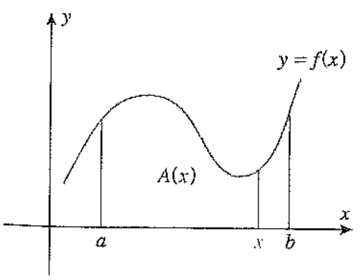
THE FUNDAMENTAL THEOREM OF CALCULUS



▲ Figure 4.6.1

As in earlier sections, let us begin by assuming that f is nonnegative and continuous on an interval $[a, b]$, in which case the area A under the graph of f over the interval $[a, b]$ is represented by the definite integral
$$A = \int_a^b f(x)\,dx \tag{1}$$
(Figure 4.6.1).

Recall that our discussion of the antiderivative method in Section 4.1 suggested that if $A(x)$ is the area under the graph of f from a to x (Figure 4.6.2), then



▲ Figure 4.6.2

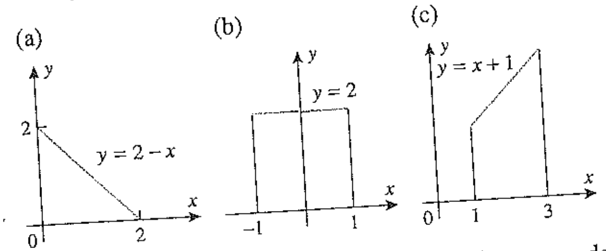
- $A'(x) = f(x)$
- $A(a) = 0$ — The area under the curve from a to a is the area above the single point a, and hence is zero.
- $A(b) = A$ — The area under the curve from a to b is A.

The formula $A'(x) = f(x)$ states that $A(x)$ is an antiderivative of $f(x)$, which implies that every other antiderivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$. Accordingly, let
$$F(x) = A(x) + C$$
be any antiderivative of $f(x)$, and consider what happens when we subtract $F(a)$ from $F(b)$:
$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) - A(a) = A - 0 = A$$

EXERCISE SET 4.6

Graphing Utility   C CAS

1. In each part, use a definite integral to find the area of the region, and check your answer using an appropriate formula from geometry.



2. In each part, use a definite integral to find the area under the curve $y = f(x)$ over the stated interval, and check your answer using an appropriate formula from geometry.
   (a) $f(x) = x;\ [0, 5]$
   (b) $f(x) = 5;\ [3, 9]$
   (c) $f(x) = x + 3;\ [-1, 2]$

3. In each part, sketch the analogue of Figure 4.6.10 for the specified region. [Let $y = f(x)$ denote the upper boundary of the region. If x^* is unique, label both it and $f(x^*)$ on your sketch. Otherwise, label $f(x^*)$ on your sketch, and determine all values of x^* that satisfy Equation (8).]
   (a) The region in part (a) of Exercise 1.
   (b) The region in part (b) of Exercise 1.
   (c) The region in part (c) of Exercise 1.

4. In each part, sketch the analogue of Figure 4.6.10 for the function and interval specified. [If x^* is unique, label both it and $f(x^*)$ on your sketch. Otherwise, label $f(x^*)$ on your sketch, and determine all values of x^* that satisfy Equation (8).]
   (a) The function and interval in part (a) of Exercise 2.
   (b) The function and interval in part (b) of Exercise 2.
   (c) The function and interval in part (c) of Exercise 2.

5–8 Find the area under the curve $y = f(x)$ over the stated interval.

5. $f(x) = x^3;\ [2, 3]$
6. $f(x) = x^4;\ [-1, 1]$
7. $f(x) = 3\sqrt{x};\ [1, 4]$
8. $f(x) = x^{-2/3};\ [1, 27]$

9–10 Find all values of x^* in the stated interval that satisfy Equation (8) in the Mean-Value Theorem for Integrals (4.6.2), and explain what these numbers represent.

9. (a) $f(x) = \sqrt{x};\ [0, 3]$
   (b) $f(x) = x^2 + x;\ [-12, 0]$
10. (a) $f(x) = \sin x;\ [-\pi, \pi]$ (b) $f(x) = 1/x^2;\ [1, 3]$

11–22 Evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus.

11. $\displaystyle\int_{-2}^{1} (x^2 - 6x + 12)\,dx$
12. $\displaystyle\int_{-1}^{2} 4x(1 - x^2)\,dx$
13. $\displaystyle\int_{1}^{4} \frac{4}{x^2}\,dx$
14. $\displaystyle\int_{1}^{2} \frac{1}{x^6}\,dx$
15. $\displaystyle\int_{4}^{9} 2x\sqrt{x}\,dx$
16. $\displaystyle\int_{1}^{4} \frac{1}{x\sqrt{x}}\,dx$
17. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta$
18. $\displaystyle\int_{0}^{\pi/4} \sec^2\theta\,d\theta$
19. $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\,dx$
20. $\displaystyle\int_{0}^{\pi/3} (2x - \sec x \tan x)\,dx$
21. $\displaystyle\int_{1}^{4} \left(\frac{1}{\sqrt{t}} - 3\sqrt{t}\right) dt$
22. $\displaystyle\int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x}\right) dx$

23–24 Use Theorem 4.5.5 to evaluate the given integrals.

23. (a) $\displaystyle\int_{-1}^{1} |2x - 1|\,dx$ (b) $\displaystyle\int_{0}^{3\pi/4} |\cos x|\,dx$
24. (a) $\displaystyle\int_{-1}^{2} \sqrt{2 + |x|}\,dx$ (b) $\displaystyle\int_{0}^{\pi/2} \left|\tfrac{1}{2} - \cos x\right| dx$

25–26 A function $f(x)$ is defined piecewise on an interval. In these exercises: (a) Use Theorem 4.5.5 to find the integral of $f(x)$ over the interval. (b) Find an antiderivative of $f(x)$ on the interval. (c) Use parts (a) and (b) to verify Part 1 of the Fundamental Theorem of Calculus.

25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ x^2, & 1 < x \le 2 \end{cases}$
26. $f(x) = \begin{cases} \sqrt{x}, & 0 \le x < 1 \\ 1/x^2, & 1 \le x \le 4 \end{cases}$

27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. There does not exist a differentiable function $F(x)$ such that $F'(x) = |x|$.
28. If $f(x)$ is continuous on the interval $[a, b]$, and if the definite integral of $f(x)$ over this interval has value 0, then the equation $f(x) = 0$ has at least one solution in the interval $[a, b]$.
29. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then
$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$
if and only if
$$G(a) + F(b) = F(a) + G(b)$$
30. If $f(x)$ is continuous everywhere and
$$F(x) = \int_0^x f(t)\,dt$$
then the equation $F(x) = 0$ has at least one solution.

31–34 Use a calculating utility to find the midpoin… mation of the integral using $n = 20$ subintervals, and th… the exact value of the integral using Part 1 of the Fundam… Theorem of Calculus.

31. $\displaystyle\int_1^3 \frac{1}{x^2}\,dx$
32. $\displaystyle\int_0^{\pi/2} \sin x\,dx$
33. $\displaystyle\int_{-1}^{1} \sec^2 x\,dx$
34. $\displaystyle\int_1^5 \frac{1}{x^3}\,dx$

35–38 Sketch the region described and find its area.

35. The region under the curve $y = x^2 + 1$ and over the interval $[0, 3]$.
36. The region below the curve $y = x - x^2$ and above the x-axis.
37. The region under the curve $y = 3\sin x$ and over the interval $[0, 2\pi/3]$.
38. The region below the interval $[-2, -1]$ and above the curve $y = x^3$.

42 Sketch the curve and find the total area between the curve and the given interval on the x-axis.

39. $y = x^2 - x$; $[0, 2]$
40. $y = \sin x$; $[0, 3\pi/2]$
41. $y = 2\sqrt{x+1} - 3$; $[0, 3]$
42. $y = \dfrac{x^2 - 1}{x^2}$; $[\frac{1}{2}, 2]$

43. A student wants to find the area enclosed by the graphs of $y = \cos x$, $y = 0$, $x = 0$, and $x = 0.8$.
   (a) Show that the exact area is $\sin 0.8$.
   (b) The student uses a calculator to approximate the result in part (a) to three decimal places and obtains an incorrect answer of 0.014. What was the student's error? Find the correct approximation.

FOCUS ON CONCEPTS

44. Explain why the Fundamental Theorem of Calculus may be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.
45. (a) If $h'(t)$ is the rate of change of a child's height measured in inches per year, what does the integral $\int_0^{10} h'(t)\,dt$ represent, and what are its units?
   (b) If $r'(t)$ is the rate of change of the radius of a spherical balloon measured in centimeters per second, what does the integral $\int_1^2 r'(t)\,dt$ represent, and what are its units?
   (c) If $H(t)$ is the rate of change of the speed of sound with respect to temperature measured in ft/s per °F, what does the integral $\int_{32}^{100} H(t)\,dt$ represent, and what are its units?
   (d) If $v(t)$ is the velocity of a particle in rectilinear motion, measured in cm/h, what does the integral $\int_{t_1}^{t_2} v(t)\,dt$ represent, and what are its units?

…$'(t)$ … …s from

47. Define …
   (a) Use Part 2 … find $F'(x)$.
   (b) Check the resu… and then differentiating.
48. Define $F(x)$ by
$$F(x) = \int_{\pi/4} \ldots t\,dt$$
   (a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.
   (b) Check the result in part (a) by first integrating and then differentiating.

49–52 Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives.

49. (a) $\displaystyle\frac{d}{dx}\int_1^x \sin(t^2)\,dt$ (b) $\displaystyle\frac{d}{dx}\int_1^x \sqrt{1-\cos t}\,dt$
50. (a) $\displaystyle\frac{d}{dx}\int_0^x \frac{dt}{1+\sqrt{t}}$ (b) $\displaystyle\frac{d}{dx}\int_2^x \frac{dt}{t^2+3t-4}$
51. $\displaystyle\frac{d}{dx}\int_x^0 t\sec t\,dt$ [*Hint:* Use Definition 4.5.3(b).]
52. $\displaystyle\frac{d}{du}\int_0^u |x|\,dx$
53. Let $F(x) = \displaystyle\int_4^x \sqrt{t^2+9}\,dt$. Find
   (a) $F(4)$ (b) $F'(4)$ (c) $F''(4)$.
54. Let $F(x) = \displaystyle\int_0^x \frac{\cos t}{t^2+3t+5}\,dt$. Find
   (a) $F(0)$ (b) $F'(0)$ (c) $F''(0)$.
55. Let $F(x) = \displaystyle\int_0^x \frac{t-3}{t^2+7}\,dt$ for $-\infty < x < +\infty$.
   (a) Find the value of x where F attains its minimum value.
   (b) Find intervals over which F is only increasing or only decreasing.
   (c) Find open intervals over which F is only concave up or only concave down.

C 56. Use the plotting and numerical integration commands of a CAS to generate the graph of the function F in Exercise 55 over the interval $-20 \le x \le 20$, and confirm that the graph is consistent with the results obtained in that exercise.

EXERCISE SET 4.8 C CAS

1. (a) Find f_{ave} of $f(x) = 2x$ over $[0, 4]$.
   (b) Find a point x^* in $[0, 4]$ such that $f(x^*) = f_{\text{ave}}$.
   (c) Sketch a graph of $f(x) = 2x$ over $[0, 4]$, and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.

2. (a) Find f_{ave} of $f(x) = x^2$ over $[0, 2]$.
   (b) Find a point x^* in $[0, 2]$ such that $f(x^*) = f_{\text{ave}}$.
   (c) Sketch a graph of $f(x) = x^2$ over $[0, 2]$, and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.

3–8. Find the average value of the function over the given interval.

3. $f(x) = 3x$; $[1, 3]$
4. $f(x) = \sqrt[3]{x}$; $[-1, 8]$
5. $f(x) = \sin x$; $[0, \pi]$
6. $f(x) = \sec x \tan x$; $[0, \pi/3]$
7. $f(x) = \dfrac{x}{(5x^2 + 1)^2}$; $[0, 2]$
8. $f(x) = \sec^2 x$; $[-\pi/4, \pi/4]$

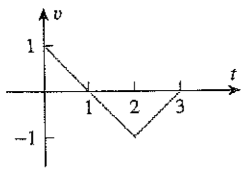
FOCUS ON CONCEPTS

9. Let $f(x) = 3x^2$.
   (a) Find the arithmetic average of the values $f(0.4)$, $f(0.8)$, $f(1.2)$, $f(1.6)$, and $f(2.0)$.
   (b) Find the arithmetic average of the values $f(0.1)$, $f(0.2)$, $f(0.3)$, $\ldots$, $f(2.0)$.
   (c) Find the average value of f on $[0, 2]$.
   (d) Explain why the answer to part (c) is less than the answers to parts (a) and (b).

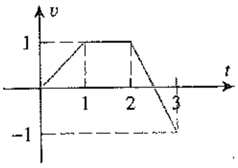
10. In parts (a)–(d), let $f(x) = 1 + \dfrac{1}{x^2}$.
   (a) Find the arithmetic average of the values $f\left(\frac{6}{5}\right)$, $f\left(\frac{7}{5}\right)$, $f\left(\frac{8}{5}\right)$, $f\left(\frac{9}{5}\right)$, and $f(2)$.
   (b) Find the arithmetic average of the values $f(1.1)$, $f(1.2)$, $f(1.3)$, $\ldots$, $f(2)$.
   (c) Find the average value of f on $[1, 2]$.
   (d) Explain why the answer to part (c) is greater than the answers to parts (a) and (b).

11. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the average velocity of the particle over the time interval $0 \le t \le 3$.

(a)



(b)



12. Suppose that a particle moving along a line starts from rest and has an average velocity of 2 ft/s over the time interval $0 \le t \le 5$. Sketch a velocity versus time curve for the particle assuming that the particle is also at rest at time $t = 5$. Explain how your curve satisfies the required properties.

13. Suppose that f is a linear function. Using the graph of f, explain why the average value of f on $[a, b]$ is
$$f\left(\frac{a+b}{2}\right)$$

14. Suppose that a particle moves along a coordinate line with constant acceleration. Show that the average velocity of the particle during a time interval $[a, b]$ matches the velocity of the particle at the midpoint of the interval.

15–18 True–False Determine whether the statement is true or false. Explain your answer. (Assume that f and g denote continuous functions on an interval $[a, b]$ and that f_{ave} and g_{ave} denote the respective average values of f and g on $[a, b]$.)

15. If $g_{\text{ave}} < f_{\text{ave}}$, then $g(x) \le f(x)$ on $[a, b]$.

16. The average value of a constant multiple of f is the same multiple of f_{ave}; that is, if c is any constant,
$$(c \cdot f)_{\text{ave}} = c \cdot f_{\text{ave}}$$

17. The average of the sum of two functions on an interval is the sum of the average values of the two functions on the interval; that is,
$$(f + g)_{\text{ave}} = f_{\text{ave}} + g_{\text{ave}}$$

18. The average of the product of two functions on an interval is the product of the average values of the two functions on the interval; that is
$$(f \cdot g)_{\text{ave}} = f_{\text{ave}} \cdot g_{\text{ave}}$$

19. (a) Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = 3t^3 + 2$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ by integrating.
   (b) Suppose that the position function of a particle moving along a coordinate line is $s(t) = 6t^2 + t$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ algebraically.

20. (a) Suppose that the acceleration function of a particle moving along a coordinate line is $a(t) = t + 1$. Find the average acceleration of the particle over the time interval $0 \le t \le 5$ by integrating.
   (b) Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = \cos t$. Find the average acceleration of the particle over the time interval $0 \le t \le \pi/4$ algebraically.

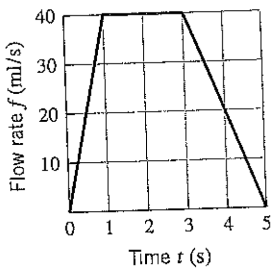
21. Water is run at a constant rate of 1 ft^3/min to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and then check your conjecture by integrating. [Take the weight density of water to be 62.4 lb/ft^3.]

22. (a) The temperature of a 10 m long metal bar is 15°C at one end and 30°C at the other end. Assuming that the temperature increases linearly from the cooler end to the hotter end, what is the average temperature of the bar?
(b) Explain why there must be a point on the bar where the temperature is the same as the average, and find it.

23. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.M. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t) = 100(1 - 0.0001t^2)$ cars per minute, where t is the time (in minutes) since 4:30 P.M. Find the average rate, in cars per minute, at which cars enter the highway during the first half-hour of rush hour.

24. Suppose that the value of a yacht in dollars after t years of use is $V(t) = 275{,}000\sqrt{\frac{20}{t+20}}$. What is the average value of the yacht over its first 10 years of use?

25. A large juice glass containing 60 ml of orange juice is replenished by a server. The accompanying figure shows the rate at which orange juice is poured into the glass in milliliters per second (ml/s). Show that the average rate of change of the volume of juice in the glass during these 5 s is equal to the average value of the rate of flow of juice into the glass.



◀ **Figure Ex-25**

C 26. The function J_0 defined by

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t)\, dt$$

is called the ***Bessel function of order zero***.
(a) Find a function f and an interval $[a, b]$ for which $J_0(1)$ is the average value of f over $[a, b]$.
(b) Estimate $J_0(1)$.
(c) Use a CAS to graph the equation $y = J_0(x)$ over the interval $0 \le x \le 8$.
(d) Estimate the smallest positive zero of J_0.

27. Find a positive value of k such that the average value of $f(x) = \sqrt{3x}$ over the interval $[0, k]$ is 6.

28. Suppose that a tumor grows at the rate of $r(t) = kt$ grams per week for some positive constant k, where t is the number of weeks since the tumor appeared. When, during the second 26 weeks of growth, is the mass of the tumor the same as its average mass during that period?

29. **Writing** Consider the following statement: *The average value of the rate of change of a function over an interval is equal to the average rate of change of the function over that interval.* Write a short paragraph that explains why this statement may be interpreted as a rewording of Part 1 of the Fundamental Theorem of Calculus.

30. **Writing** If an automobile gets an average of 25 miles per gallon of gasoline, then it is also the case that on average the automobile expends 1/25 gallon of gasoline per mile. Interpret this statement using the concept of the average value of a function over an interval.

✔ QUICK CHECK ANSWERS 4.8

1. $\frac{1}{n}\sum_{k=1}^{n} a_k$ 2. $\frac{1}{b-a}\int_a^b f(x)\, dx$ 3. $f(x^*)$ 4. 40

31. $\displaystyle\int_{\pi/12}^{\pi/9} \sec^2 3\theta\, d\theta$ **32.** $\displaystyle\int_{\pi/6}^{\pi/3} \csc^2 2\theta\, d\theta$

33. $\displaystyle\int_0^1 \frac{y^2\, dy}{\sqrt{4-3y}}$ **34.** $\displaystyle\int_{-1}^4 \frac{x\, dx}{\sqrt{5+x}}$

35. (a) Use a CAS to find the exact value of the integral

$$\int_0^{\pi/6} \sin^4 x \cos^3 x\, dx$$

(b) Confirm the exact value by hand calculation. [*Hint:* Use the identity $\cos^2 x = 1 - \sin^2 x$.]

36. (a) Use a CAS to find the exact value of the integral

$$\int_{-\pi/4}^{\pi/4} \tan^4 x\, dx$$

(b) Confirm the exact value by hand calculation. [*Hint:* Use the identity $1 + \tan^2 x = \sec^2 x$.]

37. (a) Find $\displaystyle\int_0^1 f(3x+1)\, dx$ if $\displaystyle\int_1^4 f(x)\, dx = 5$.

(b) Find $\displaystyle\int_0^3 f(3x)\, dx$ if $\displaystyle\int_0^9 f(x)\, dx = 5$.

(c) Find $\displaystyle\int_{-2}^0 xf(x^2)\, dx$ if $\displaystyle\int_0^4 f(x)\, dx = 1$.

38. Given that m and n are positive integers, show that

$$\int_0^1 x^m(1-x)^n\, dx = \int_0^1 x^n(1-x)^m\, dx$$

by making a substitution. Do not attempt to evaluate the integrals.

39. Given that n is a positive integer, show that

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$$

by using a trigonometric identity and making a substitution. Do not attempt to evaluate the integrals.

40. Given that n is a positive integer, evaluate the integral

$$\int_0^1 x(1-x)^n\, dx$$

41. (a) The accompanying table shows the fraction of the Moon that is illuminated (as seen from Earth) at midnight (Eastern Standard Time) for the first week of 2005. Find the average fraction of the Moon illuminated during the first week of 2005.

Source: Data from the U.S Naval Observatory Astronomical Applications Department.

(b) The function $f(x) = 0.5 + 0.5\sin(0.213x + 2.481)$ models data for illumination of the Moon for the first 60 days of 2005. Find the average value of this illumination function over the interval $[0, 7]$.

DAY	1	2	3	4	5	6	7
ILLUMINATION	0.74	0.65	0.56	0.45	0.35	0.25	0.16

▲ **Table Ex-41**

42. Electricity is supplied to homes in the form of ***alternating current***, which means that the voltage has a sinusoidal waveform described by an equation of the form

$$V = V_p \sin(2\pi f t)$$

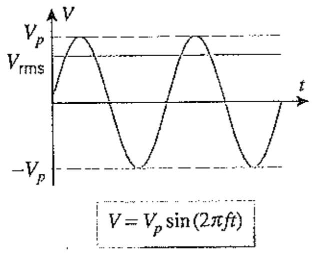
(see the accompanying figure). In this equation, V_p is called the ***peak voltage*** or ***amplitude*** of the current, f is called its ***frequency***, and $1/f$ is called its ***period***. The voltages V and V_p are measured in volts (V), the time t is measured in seconds (s), and the frequency is measured in hertz (Hz). (1 Hz = 1 cycle per second; a ***cycle*** is the electrical term for one period of the waveform.) Most alternating-current voltmeters read what is called the ***rms*** or ***root-mean-square*** value of V. By definition, this is the square root of the average value of V^2 over one period.

(a) Show that

$$V_{\text{rms}} = \frac{V_p}{\sqrt{2}}$$

[*Hint:* Compute the average over the cycle from $t = 0$ to $t = 1/f$, and use the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to help evaluate the integral.]

(b) In the United States, electrical outlets supply alternating current with an rms voltage of 120 V at a frequency of 60 Hz. What is the peak voltage at such an outlet?



◀ **Figure Ex-42**

C **43.** (a) Find the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{\sin(k\pi/n)}{n}$$

by evaluating an appropriate definite integral over the interval $[0, 1]$.

(b) Check your answer to part (a) by evaluating the limit directly with a CAS.

FOCUS ON CONCEPTS

44. Let

$$I = \int_{-1}^{1} \frac{1}{1+x^2}\, dx$$

(a) Explain why $I > 0$.

(b) Show that the substitution $x = 1/u$ results in

$$I = -\int_{-1}^{1} \frac{1}{1+x^2}\, dx = -I$$

Thus, $2I = 0$, which implies that $I = 0$. But this contradicts part (a). What is the error?

45. (a) Prove that if f is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

and give a geometric explanation of this result. [*Hint:* One way to prove that a quantity q is zero is to show that $q = -q$.]

(b) Prove that if f is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

and give a geometric explanation of this result. [*Hint:* Split the interval of integration from $-a$ to a into two parts at 0.]

46. Show that if f and g are continuous functions, then

$$\int_0^t f(t-x)g(x)\,dx = \int_0^t f(x)g(t-x)\,dx$$

47. (a) Let

$$I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$$

Show that $I = a/2$.

[*Hint:* Let $u = a - x$, and then note the difference between the resulting integrand and 1.]

(b) Use the result of part (a) to find

$$\int_0^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}\,dx$$

(c) Use the result of part (a) to find

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$$

48. Evaluate

(a) $\displaystyle\int_{-1}^{1} x\sqrt{\cos(x^2)}\,dx$

(b) $\displaystyle\int_0^{\pi} \sin^8 x \cos^5 x\,dx.$

[*Hint:* Use the substitution $u = x - (\pi/2)$.]

49. Writing The two substitution methods discussed in this section yield the same result when used to evaluate a definite integral. Write a short paragraph that carefully explains why this is the case.

50. Writing In some cases, the second method for the evaluation of definite integrals has distinct advantages over the first. Provide some illustrations, and write a short paragraph that discusses the advantages of the second method in each case. [*Hint:* To get started, consider the results in Exercises 38–40, 45, and 47.]

✔ QUICK CHECK ANSWERS 4.9

1. $F(g(b)) - F(g(a))$ 2. (a) $\displaystyle\int_1^9 u^3\,du$ (b) $\displaystyle\int_1^5 \frac{1}{2\sqrt{u}}\,du$ 3. (a) $\frac{2}{3}$ (b) $\frac{3}{4}$

CHAPTER 4 REVIEW EXERCISES

Graphing Utility C CAS

1–4 Evaluate the integrals.

1. $\displaystyle\int\left[\frac{1}{2x^3}+4\sqrt{x}\right]dx$ **2.** $\displaystyle\int [u^3 - 2u + 7]\,du$

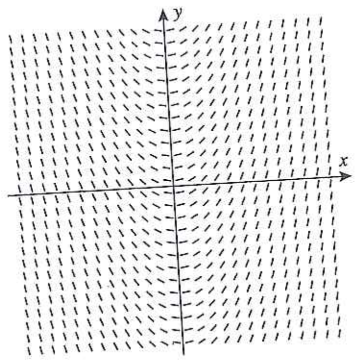
3. $\displaystyle\int [4\sin x + 2\cos x]\,dx$ **4.** $\displaystyle\int \sec x(\tan x + \cos x)\,dx$

5. Solve the initial-value problems.

(a) $\dfrac{dy}{dx} = \dfrac{1-x}{\sqrt{x}},\ y(1) = 0$

(b) $\dfrac{dy}{dx} = \cos x - 5x,\ y(0) = 1$

(c) $\dfrac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$



◀ **Figure Ex-6**

6. The accompanying figure shows the slope field for a differential equation $dy/dx = f(x)$. Which of the following functions is most likely to be $f(x)$?

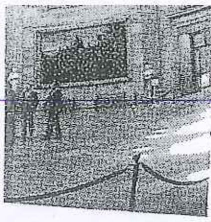
$$\sqrt{x},\quad \sin x,\quad x^4,\quad x$$

Explain your reasoning.

7. (a) Show that the substitutions $u = \sec x$ and $u = \tan x$ produce different values for the integral

$$\int \sec^2 x \tan x\,dx$$

(b) Explain why both are correct.



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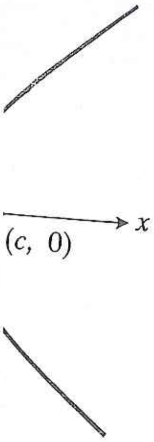
Like the parabola, ellipses and hyperbolas have reflective properties that are important in science and engineering. If an ellipse is revolved about its major axis to generate a surface (the surface is called an *ellipsoid*), and the interior is silvered to produce a mirror, light from one focus will be reflected to the other focus (Fig. 10.12). Ellipsoids reflect sound the same way, and this property is used to construct *whispering galleries*, rooms in which a person standing at one focus can hear a whisper from the other focus. Ellipsoids also appear in instruments used to study aircraft noise in wind tunnels (sound at one focus can be received at the other focus with relatively little interference from other sources).

Light directed toward one focus of a hyperbolic mirror is reflected toward the other focus (Fig. 10.13). This property of hyperbolas is combined with the reflective properties of parabolas and ellipses in designing modern telescopes (Fig. 10.14)



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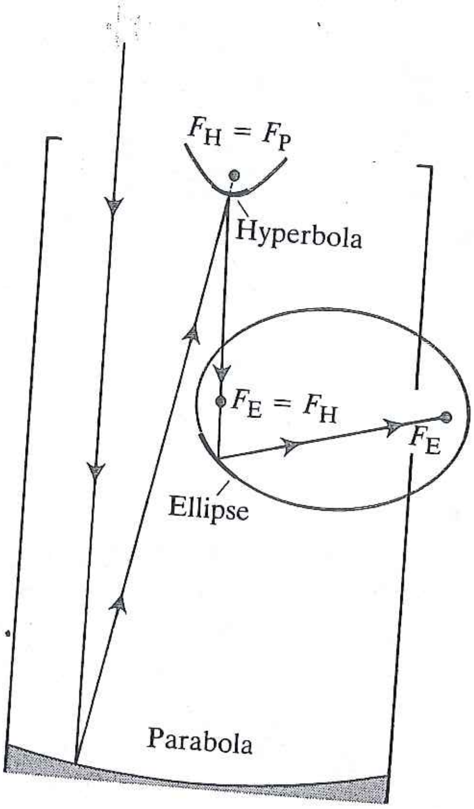

10.14 In this schematic drawing of a reflecting telescope, starlight reflects off a primary parabolic mirror toward the mirror's focus F_P. It is then reflected by a small hyperbolic mirror, whose focus is $F_H = F_P$, toward the second focus of the hyperbola, $F_E = F_H$. Since this focus is shared by an ellipse, the light is reflected by the elliptical mirror to the ellipse's second focus to be seen by an observer.