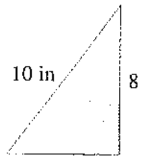
MA1123 Assignment5

[due Monday 16th November 2015]

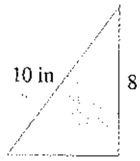
1. Page 234-237 Numbers 10,14,30,34,44,46,56,62
2. Page 251 Number 24
3. Page 257-258 Numbers 11-14,15-17,22,26.

6. A rectangle is to be inscribed in a right triangle having sides of length 6 in, 8 in, and 10 in. Find the dimensions of the rectangle with greatest area assuming the rectangle is positioned as in Figure Ex-6.

7. Solve the problem in Exercise 6 assuming the rectangle is positioned as in Figure Ex-7.



▲ Figure Ex-6



▲ Figure Ex-7

8. A rectangle has its two lower corners on the x-axis and its two upper corners on the curve $y = 16 - x^2$. For all such rectangles, what are the dimensions of the one with largest area?

9. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10.

10. Find the point P in the first quadrant on the curve $y = x^{-2}$ such that a rectangle with sides on the coordinate axes and a vertex at P has the smallest possible perimeter.

11. A rectangular area of 3200 ft^2 is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle of least cost.

12. Show that among all rectangles with perimeter p, the square has the maximum area.

13. Show that among all rectangles with area A, the square has the minimum perimeter.

14. A wire of length 12 in can be bent into a circle, bent into a square, or cut into two pieces to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be
(a) a maximum (b) a minimum?

15. A rectangle R in the plane has corners at $(\pm 8, \pm 12)$, and a 100 by 100 square S is positioned in the plane so that its sides are parallel to the coordinate axes and the lower left corner of S is on the line $y = -3x$. What is the largest possible area of a region in the plane that is contained in both R and S?

16. Solve the problem in Exercise 15 if S is a 16 by 16 square.

17. Solve the problem in Exercise 15 if S is positioned with its lower left corner on the line $y = -6x$.

18. A rectangular page is to contain 42 square inches of printable area. The margins at the top and bottom of the page are each 1 inch, one side margin is 1 inch, and the other side margin is 2 inches. What should the dimensions of the page be so that the least amount of paper is used?

19. A box with a square base is taller than it is wide. In order to send the box through the U.S. mail, the height of the box and the perimeter of the base can sum to no more than 108 in. What is the maximum volume for such a box?

20. A box with a square base is wider than it is tall. In order to send the box through the U.S. mail, the width of the box and the perimeter of one of the (nonsquare) sides of the box can sum to no more than 108 in. What is the maximum volume for such a box?

21. An open box is to be made from a 3 ft by 8 ft rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum volume that the box can have.

22. A closed rectangular container with a square base is to have a volume of 2250 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$3 per in^2. Find the dimensions of the container of least cost.

23. A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.

24. A container with square base, vertical sides, and open top is to be made from 1000 ft^2 of material. Find the dimensions of the container with greatest volume.

25. A rectangular container with two square sides and an open top is to have a volume of V cubic units. Find the dimensions of the container with minimum surface area.

26. A church window consisting of a rectangle topped by a semicircle is to have a perimeter p. Find the radius of the semicircle if the area of the window is to be maximum.

27. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

28. Find the dimensions of the right circular cylinder of greatest surface area that can be inscribed in a sphere of radius R.

29. A closed, cylindrical can is to have a volume of V cubic units. Show that the can of minimum surface area is achieved when the height is equal to the diameter of the base.

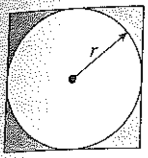
30. A closed cylindrical can is to have a surface area of S square units. Show that the can of maximum volume is achieved when the height is equal to the diameter of the base.

31. A cylindrical can, open at the top, is to hold 500 cm^3 of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

32. A soup can in the shape of a right circular cylinder of radius r and height h is to have a prescribed volume V. The top and bottom are cut from squares as shown in Figure Ex-32 on the next page. If the shaded corners are wasted, but there is no other waste, find the ratio r/h for the can requiring the least material (including waste).

33. A box-shaped wire frame consists of two identical wire squares whose vertices are connected by four straight wires of equal length (Figure Ex-33 on the next page). If the

frame is to be made from a wire of length L, what should the dimensions be to obtain a box of greatest volume?



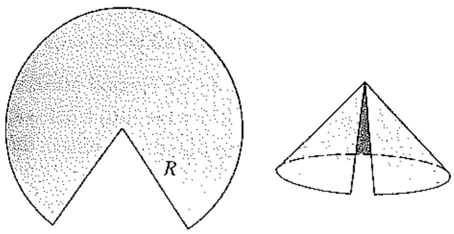
▲ Figure Ex-32

▲ Figure Ex-33

34. Suppose that the sum of the surface areas of a sphere and a cube is a constant.
   (a) Show that the sum of their volumes is smallest when the diameter of the sphere is equal to the length of an edge of the cube.
   (b) When will the sum of their volumes be greatest?

35. Find the height and radius of the cone of slant height L whose volume is as large as possible.

36. A cone is made from a circular sheet of radius R by cutting out a sector and gluing the cut edges of the remaining piece together (Figure Ex-36). What is the maximum volume attainable for the cone?



▲ Figure Ex-36

37. A cone-shaped paper drinking cup is to hold 100 cm^3 of water. Find the height and radius of the cup that will require the least amount of paper.

38. Find the dimensions of the isosceles triangle of least area that can be circumscribed about a circle of radius R.

39. Find the height and radius of the right circular cone with least volume that can be circumscribed about a sphere of radius R.

40. A commercial cattle ranch currently allows 20 steers per acre of grazing land; on the average its steers weigh 2000 lb at market. Estimates by the Agriculture Department indicate that the average market weight per steer will be reduced by 50 lb for each additional steer added per acre of grazing land. How many steers per acre should be allowed in order for the ranch to get the largest possible total market weight for its cattle?

41. A company mines low-grade nickel ore. If the company mines x tons of ore, it can sell the ore for $p = 225 - 0.25x$ dollars per ton. Find the revenue and marginal revenue functions. At what level of production would the company obtain the maximum revenue?

42. A fertilizer producer finds that it can sell its product at a price of $p = 300 - 0.1x$ dollars per unit when it produces x units of fertilizer. The total production cost (in dollars) for x units is
$$C(x) = 15{,}000 + 125x + 0.025x^2$$
If the production capacity of the firm is at most 1000 units of fertilizer in a specified time, how many units must be manufactured and sold in that time to maximize the profit?

43. (a) A chemical manufacturer sells sulfuric acid in bulk at a price of \$100 per unit. If the daily total production cost in dollars for x units is
$$C(x) = 100{,}000 + 50x + 0.0025x^2$$
and if the daily production capacity is at most 7000 units, how many units of sulfuric acid must be manufactured and sold daily to maximize the profit?
   (b) Would it benefit the manufacturer to expand the daily production capacity?
   (c) Use marginal analysis to approximate the effect on profit if daily production could be increased from 7000 to 7001 units.

44. A firm determines that x units of its product can be sold daily at p dollars per unit, where
$$x = 1000 - p$$
The cost of producing x units per day is
$$C(x) = 3000 + 20x$$
   (a) Find the revenue function $R(x)$.
   (b) Find the profit function $P(x)$.
   (c) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.
   (d) Find the maximum profit.
   (e) What price per unit must be charged to obtain the maximum profit?

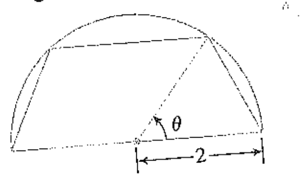
45. In a certain chemical manufacturing process, the daily weight y of defective chemical output depends on the total weight x of all output according to the empirical formula
$$y = 0.01x + 0.00003x^2$$
where x and y are in pounds. If the profit is \$100 per pound of nondefective chemical produced and the loss is \$20 per pound of defective chemical produced, how many pounds of chemical should be produced daily to maximize the total daily profit?

46. An independent truck driver charges a client \$15 for each hour of driving, plus the cost of fuel. At highway speeds of v miles per hour, the trucker's rig gets $10 - 0.07v$ miles per gallon of diesel fuel. If diesel fuel costs \$2.50 per gallon, what speed v will minimize the cost to the client?

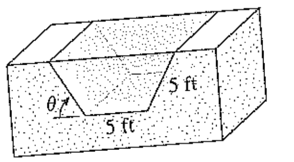
47. A trapezoid is inscribed in a semicircle of radius 2 so that one side is along the diameter (Figure Ex-47 on the next page). Find the maximum possible area for the trapezoid. [*Hint:* Express the area of the trapezoid in terms of θ.]

48. A drainage channel is to be made so that its cross section is a trapezoid with equally sloping sides (Figure Ex-48 on the next page). If the sides and bottom all have a length of 5 ft,

how should the angle θ $(0 \le \theta \le \pi/2)$ be chosen to yield the greatest cross-sectional area of the channel?



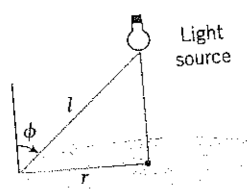
▲ Figure Ex-47



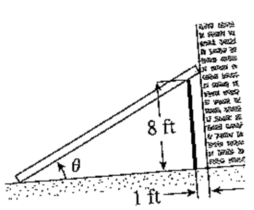
▲ Figure Ex-48

49. A lamp is suspended above the center of a round table of radius r. How high above the table should the lamp be placed to achieve maximum illumination at the edge of the table? [Assume that the illumination I is directly proportional to the cosine of the angle of incidence ϕ of the light rays and inversely proportional to the square of the distance l from the light source (Figure Ex-49).]

50. A plank is used to reach over a fence 8 ft high to support a wall that is 1 ft behind the fence (Figure Ex-50). What is the length of the shortest plank that can be used? [*Hint:* Express the length of the plank in terms of the angle θ shown in the figure.]



▲ Figure Ex-49



▲ Figure Ex-50

51. Find the coordinates of the point P on the curve

$$y = \frac{1}{x^2} \quad (x > 0)$$

where the segment of the tangent line at P that is cut off by the coordinate axes has its shortest length.

52. Find the x-coordinate of the point P on the parabola

$$y = 1 - x^2 \quad (0 < x \le 1)$$

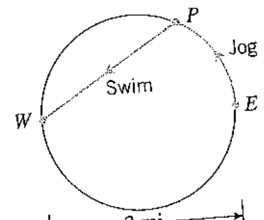
where the triangle that is enclosed by the tangent line at P and the coordinate axes has the smallest area.

53. Where on the curve $y = (1 + x^2)^{-1}$ does the tangent line have the greatest slope?

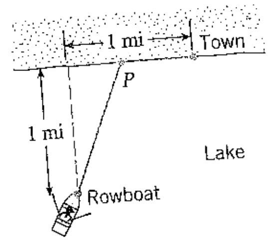
54. A rectangular water tank has a base of area four square meters. Water flows into the tank until the amount of water in the tank is 20 m^3, at which point any additional flow into the tank is diverted by an overflow valve. Suppose that the tank is initially empty, and water is pumped into the tank so that after t minutes, $(2t^3 + 7t)/(t^2 + 12)$ cubic meters of water has been pumped into the tank. At what time is the height of the water in the tank increasing most rapidly?

55. The shoreline of Circle Lake is a circle with diameter 2 mi. Nancy's training routine begins at point E on the eastern shore of the lake. She jogs along the north shore to a point P and then swims the straight line distance, if any, from P to point W diametrically opposite E (Figure Ex-55). Nancy swims at a rate of 2 mi/h and jogs at 8 mi/h. How far should Nancy jog in order to complete her training routine in
(a) the least amount of time
(b) the greatest amount of time?

56. A man is floating in a rowboat 1 mile from the (straight) shoreline of a large lake. A town is located on the shoreline 1 mile from the point on the shoreline closest to the man. As suggested in Figure Ex-56, he intends to row in a straight line to some point P on the shoreline and then walk the remaining distance to the town. To what point should he row in order to reach his destination in the least time if
(a) he can walk 5 mi/h and row 3 mi/h
(b) he can walk 5 mi/h and row 4 mi/h?



▲ Figure Ex-55

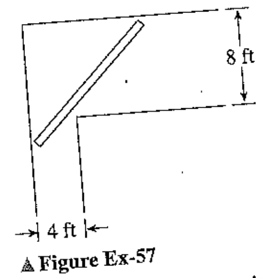


▲ Figure Ex-56

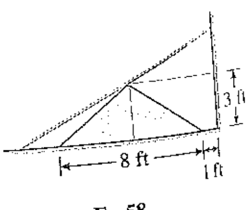
57. A pipe of negligible diameter is to be carried horizontally around a corner from a hallway 8 ft wide into a hallway 4 ft wide (Figure Ex-57). What is the maximum length that the pipe can have?

Source: An interesting discussion of this problem in the case where the diameter of the pipe is not neglected is given by Norman Miller in the *American Mathematical Monthly*, Vol. 56, 1949, pp. 177–179.

58. A concrete barrier whose cross section is an isosceles triangle runs parallel to a wall. The height of the barrier is 3 ft, the width of the base of a cross section is 8 ft, and the barrier is positioned on level ground with its base 1 ft from the wall. A straight, stiff metal rod of negligible diameter has one end on the ground, the other end against the wall, and touches the top of the barrier (Figure Ex-58). What is the minimum length the rod can have?



▲ Figure Ex-57



▲ Figure Ex-58

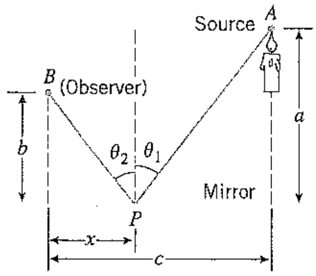
59. Suppose that the intensity of a point light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two point light sources with strengths of S and $8S$ are separated by a distance of 90 cm. Where on the line segment between the two sources is the total intensity a minimum?

60. Given points $A(2, 1)$ and $B(5, 4)$, find the point P in the interval $[2, 5]$ on the x-axis that maximizes angle APB.

61. The lower edge of a painting, 10 ft in height, is 2 ft above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?

FOCUS ON CONCEPTS

62. ***Fermat's principle*** (biography on p. 225) in optics states that light traveling from one point to another follows that path for which the total travel time is minimum. In a uniform medium, the paths of "minimum time" and "shortest distance" turn out to be the same, so that light, if unobstructed, travels along a straight line. Assume that we have a light source, a flat mirror, and an observer in a uniform medium. If a light ray leaves the source, bounces off the mirror, and travels on to the observer, then its path will consist of two line segments, as shown in Figure Ex-62. According to Fermat's principle, the path will be such that the total travel time t is minimum or, since the medium is uniform, the path will be such that the total distance traveled from A to P to B is as small as possible. Assuming the minimum occurs when $dt/dx = 0$, show that the light ray will strike the mirror at the point P where the "angle of incidence" θ_1 equals the "angle of reflection" θ_2.



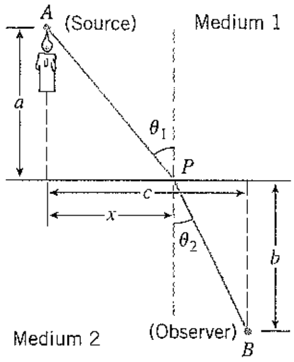
◀ **Figure Ex-62**

63. Fermat's principle (Exercise 62) also explains why light rays traveling between air and water undergo bending (refraction). Imagine that we have two uniform media (such as air and water) and a light ray traveling from a source A in one medium to an observer B in the other medium (Figure Ex-63). It is known that light travels at a constant speed in a uniform medium, but more slowly in a dense medium (such as water) than in a thin medium (such as air). Consequently, the path of shortest time from A to B is not necessarily a straight line, but rather some broken line path A to P to B allowing the light to take greatest advantage of its higher speed through the thin medium. ***Snell's law of refraction*** (biography on p. 238) states that the path of the light ray will be such that

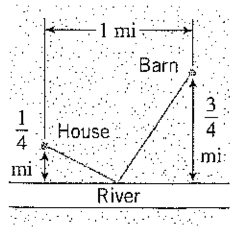
$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

where v_1 is the speed of light in the first medium, v_2 is the speed of light in the second medium, and θ_1 and θ_2 are the angles shown in Figure Ex-63. Show that this follows from the assumption that the path of minimum time occurs when $dt/dx = 0$.

64. A farmer wants to walk at a constant rate from her barn to a straight river, fill her pail, and carry it to her house in the least time.
   (a) Explain how this problem relates to Fermat's principle and the light-reflection problem in Exercise 62.
   (b) Use the result of Exercise 62 to describe geometrically the best path for the farmer to take.
   (c) Use part (b) to determine where the farmer should fill her pail if her house and barn are located as in Figure Ex-64.



▲ **Figure Ex-63**



▲ **Figure Ex-64**

65. If an unknown physical quantity x is measured n times, the measurements $x_1, x_2, \ldots, x_n$ often vary because of uncontrollable factors such as temperature, atmospheric pressure, and so forth. Thus, a scientist is often faced with the problem of using n different observed measurements to obtain an estimate $\bar{x}$ of an unknown quantity x. One method for making such an estimate is based on the ***least squares principle***, which states that the estimate $\bar{x}$ should be chosen to minimize

$$s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2$$

which is the sum of the squares of the deviations between the estimate $\bar{x}$ and the measured values. Show that the estimate resulting from the least squares principle is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

that is, $\bar{x}$ is the arithmetic average of the observed values.

66. Prove: If $f(x) \geq 0$ on an interval and if $f(x)$ has a maximum value on that interval at x_0, then $\sqrt{f(x)}$ also has a maximum value at x_0. Similarly for minimum values. [*Hint:* Use the fact that $\sqrt{x}$ is an increasing function on the interval $[0, +\infty)$.]

67. **Writing** Discuss the importance of finding intervals of possible values imposed by physical restrictions on variables in an applied maximum or minimum problem.

23. The *mechanic's rule* for approximating square roots states that $\sqrt{a} \approx x_{n+1}$, where
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3, \ldots$$
and x_1 is any positive approximation to $\sqrt{a}$.
(a) Apply Newton's Method to
$$f(x) = x^2 - a$$
to derive the mechanic's rule.
(b) Use the mechanic's rule to approximate $\sqrt{10}$.

24. Many calculators compute reciprocals using the approximation $1/a \approx x_{n+1}$, where
$$x_{n+1} = x_n(2 - ax_n), \quad n = 1, 2, 3, \ldots$$
and x_1 is an initial approximation to $1/a$. This formula makes it possible to perform divisions using multiplications and subtractions, which is a faster procedure than dividing directly.
(a) Apply Newton's Method to
$$f(x) = \frac{1}{x} - a$$
to derive this approximation.
(b) Use the formula to approximate $\frac{1}{17}$.

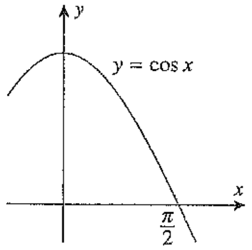
25. Use Newton's Method to approximate the absolute minimum of $f(x) = \frac{1}{4}x^4 + x^2 - 5x$.

26. Use Newton's Method to approximate the absolute maximum of $f(x) = x \sin x$ on the interval $[0, \pi]$.

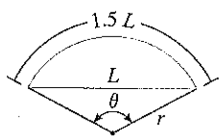
27. Use Newton's Method to approximate the coordinates of the the point on the parabola $y = x^2$ that is closest to the point $(1, 0)$.

28. Use Newton's Method to approximate the dimensions of the rectangle of largest area that can be inscribed under the curve $y = \cos x$ for $0 \le x \le \pi/2$ (Figure Ex-28).

29. (a) Show that on a circle of radius r, the central angle θ that subtends an arc whose length is 1.5 times the length L of its chord satisfies the equation $\theta = 3\sin(\theta/2)$ (Figure Ex-29).
(b) Use Newton's Method to approximate θ.

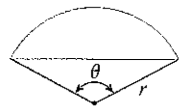


▲ Figure Ex-28



▲ Figure Ex-29

30. A *segment* of a circle is the region enclosed by an arc and its chord (Figure Ex-34). If r is the radius of the circle and θ the angle subtended at the center of the circle, then it can be shown that the area A of the segment is $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians. Find the value of θ for which the area of the segment is one-fourth the area of the circle. Give θ to the nearest degree.



◄ Figure Ex-30

31–32 Use Newton's Method to approximate all real values of y satisfying the given equation for the indicated value of x.

31. $xy^4 + x^3y = 1;\ x = 1$ 32. $xy - \cos\left(\frac{1}{2}xy\right) = 0;\ x = 2$

33. An ***annuity*** is a sequence of equal payments that are paid or received at regular time intervals. For example, you may want to deposit equal amounts at the end of each year into an interest-bearing account for the purpose of accumulating a lump sum at some future time. If, at the end of each year, interest of $i \times 100\%$ on the account balance for that year is added to the account, then the account is said to pay $i \times 100\%$ interest, ***compounded annually***. It can be shown that if payments of Q dollars are deposited at the end of each year into an account that pays $i \times 100\%$ compounded annually, then at the time when the nth payment and the accrued interest for the past year are deposited, the amount $S(n)$ in the account is given by the formula
$$S(n) = \frac{Q}{i}[(1+i)^n - 1]$$
Suppose that you can invest \$5000 in an interest-bearing account at the end of each year, and your objective is to have \$250,000 on the 25th payment. Approximately what annual compound interest rate must the account pay for you to achieve your goal? [*Hint:* Show that the interest rate i satisfies the equation $50i = (1+i)^{25} - 1$, and solve it using Newton's Method.]

FOCUS ON CONCEPTS

34. (a) Use a graphing utility to generate the graph of
$$f(x) = \frac{x}{x^2+1}$$
and use it to explain what happens if you apply Newton's Method with a starting value of $x_1 = 2$. Check your conclusion by computing x_2, x_3, x_4, and x_5.
(b) Use the graph generated in part (a) to explain what happens if you apply Newton's Method with a starting value of $x_1 = 0.5$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

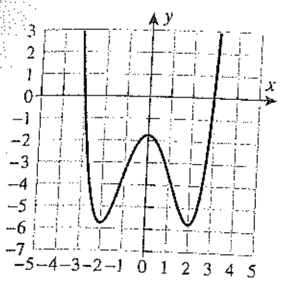
35. (a) Apply Newton's Method to $f(x) = x^2 + 1$ with a starting value of $x_1 = 0.5$, and determine if the values of $x_2, \ldots, x_{10}$ appear to converge.
(b) Explain what is happening.

36. In each part, explain what happens if you apply Newton's Method to a function f when the given condition is satisfied for some value of n.
(a) $f(x_n) = 0$ (b) $x_{n+1} = x_n$
(c) $x_{n+2} = x_n \ne x_{n+1}$

✔ QUICK CHECK EXERCISES 3.8

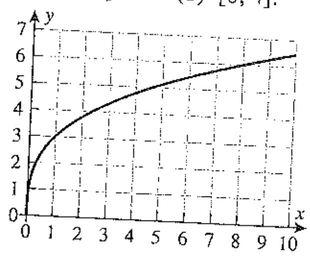
(See page 259 for answers.)

1. Let $f(x) = x^2 - x$.
   (a) An interval on which f satisfies the hypotheses of Rolle's Theorem is ______.
   (b) Find all values of c that satisfy the conclusion of Rolle's Theorem for the function f on the interval in part (a).
2. Use the accompanying graph of f to find an interval $[a, b]$ on which Rolle's Theorem applies, and find all values of c in that interval that satisfy the conclusion of the theorem.



◀ Figure Ex-2

3. Let $f(x) = x^2 - x$.
   (a) Find a point b such that the slope of the secant line through $(0, 0)$ and $(b, f(b))$ is 1.
   (b) Find all values of c that satisfy the conclusion of the Mean-Value Theorem for the function f on the interval $[0, b]$, where b is the point found in part (a).
4. Use the graph of f in the accompanying figure to estimate all values of c that satisfy the conclusion of the Mean-Value Theorem on the interval
   (a) $[0, 8]$ (b) $[0, 4]$.



◀ Figure Ex-4

5. Find a function f such that the graph of f contains the point $(1, 5)$ and such that for every value of x_0 the tangent line to the graph of f at x_0 is parallel to the tangent line to the graph of $y = x^2$ at x_0.

EXERCISE SET 3.8 Graphing Utility

1–4 Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

1. $f(x) = x^2 - 8x + 15$; $[3, 5]$
2. $f(x) = x^3 - 3x^2 + 2x$; $[0, 2]$
3. $f(x) = \cos x$; $[\pi/2, 3\pi/2]$
4. $f(x) = (x^2 - 1)/(x - 2)$; $[-1, 1]$

5–8 Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

5. $f(x) = x^2 - x$; $[-3, 5]$
6. $f(x) = x^3 + x - 4$; $[-1, 2]$
7. $f(x) = \sqrt{x + 1}$; $[0, 3]$
8. $f(x) = x - \dfrac{1}{x}$; $[3, 4]$
9. (a) Find an interval $[a, b]$ on which
$$f(x) = x^4 + x^3 - x^2 + x - 2$$
satisfies the hypotheses of Rolle's Theorem.
   (b) Generate the graph of $f'(x)$, and use it to make rough estimates of all values of c in the interval obtained in part (a) that satisfy the conclusion of Rolle's Theorem.
   (c) Use Newton's Method to improve on the rough estimates obtained in part (b).
10. Let $f(x) = x^3 - 4x$.
   (a) Find the equation of the secant line through the points $(-2, f(-2))$ and $(1, f(1))$.
   (b) Show that there is only one point c in the interval $(-2, 1)$ that satisfies the conclusion of the Mean-Value Theorem for the secant line in part (a).
   (c) Find the equation of the tangent line to the graph of f at the point $(c, f(c))$.
   (d) Use a graphing utility to generate the secant line in part (a) and the tangent line in part (c) in the same coordinate system, and confirm visually that the two lines seem parallel.

11–14 True–False Determine whether the statement is true or false. Explain your answer.

11. Rolle's Theorem says that if f is a continuous function on $[a, b]$ and $f(a) = f(b)$, then there is a point between a and b at which the curve $y = f(x)$ has a horizontal tangent line.
12. If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is a point between a and b at which the instantaneous rate of change of f matches the average rate of change of f over $[a, b]$.
13. The Constant Difference Theorem says that if two functions have derivatives that differ by a constant on an interval, then the functions are equal on the interval.

14. One application of the Mean-Value Theorem is to prove that a function with positive derivative on an interval must be increasing on that interval.

FOCUS ON CONCEPTS

15. Let $f(x) = \tan x$.
   (a) Show that there is no point c in the interval $(0, \pi)$ such that $f'(c) = 0$, even though $f(0) = f(\pi) = 0$.
   (b) Explain why the result in part (a) does not contradict Rolle's Theorem.

16. Let $f(x) = x^{2/3}$, $a = -1$, and $b = 8$.
   (a) Show that there is no point c in (a, b) such that
   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$
   (b) Explain why the result in part (a) does not contradict the Mean-Value Theorem.

17. (a) Show that if f is differentiable on $(-\infty, +\infty)$, and if $y = f(x)$ and $y = f'(x)$ are graphed in the same coordinate system, then between any two x-intercepts of f there is at least one x-intercept of f'.
   (b) Give some examples that illustrate this.

18. Review Formulas (8) and (9) in Section 2.1 and use the Mean-Value Theorem to show that if f is differentiable on $(-\infty, +\infty)$, then for any interval $[x_0, x_1]$ there is at least one point in (x_0, x_1) where the instantaneous rate of change of y with respect to x is equal to the average rate of change over the interval.

19–21 Use the result of Exercise 18 in these exercises.

19. An automobile travels 4 mi along a straight road in 5 min. Show that the speedometer reads exactly 48 mi/h at least once during the trip.

20. At 11 A.M. on a certain morning the outside temperature was 76°F. At 11 P.M. that evening it had dropped to 52°F.
   (a) Show that at some instant during this period the temperature was decreasing at the rate of 2°F/h.
   (b) Suppose that you know the temperature reached a high of 88°F sometime between 11 A.M. and 11 P.M. Show that at some instant during this period the temperature was decreasing at a rate greater than 3°F/h.

21. Suppose that two runners in a 100 m dash finish in a tie. Show that they had the same velocity at least once during the race.

22. Use the fact that
$$\frac{d}{dx}(3x^4 + x^2 - 4x) = 12x^3 + 2x - 4$$
to show that the equation $12x^3 + 2x - 4 = 0$ has at least one solution in the interval $(0, 1)$.

23. (a) Use the Constant Difference Theorem (3.8.3) to show that if $f'(x) = g'(x)$ for all x in the interval $(-\infty, +\infty)$, and if f and g have the same value at some point x_0, then $f(x) = g(x)$ for all x in $(-\infty, +\infty)$.
   (b) Use the result in part (a) to confirm the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

24. (a) Use the Constant Difference Theorem (3.8.3) to show that if $f'(x) = g'(x)$ for all x in $(-\infty, +\infty)$, and if $f(x_0) - g(x_0) = c$ at some point x_0, then
   $$f(x) - g(x) = c$$
   for all x in $(-\infty, +\infty)$.
   (b) Use the result in part (a) to show that the function
   $$h(x) = (x - 1)^3 - (x^2 + 3)(x - 3)$$
   is constant for all x in $(-\infty, +\infty)$, and find the constant.
   (c) Check the result in part (b) by multiplying out and simplifying the formula for $h(x)$.

FOCUS ON CONCEPTS

25. (a) Use the Mean-Value Theorem to show that if f is differentiable on an interval, and if $|f'(x)| \le M$ for all values of x in the interval, then
   $$|f(x) - f(y)| \le M|x - y|$$
   for all values of x and y in the interval.
   (b) Use the result in part (a) to show that
   $$|\sin x - \sin y| \le |x - y|$$
   for all real values of x and y.

26. (a) Use the Mean-Value Theorem to show that if f is differentiable on an open interval, and if $|f'(x)| \ge M$ for all values of x in the interval, then
   $$|f(x) - f(y)| \ge M|x - y|$$
   for all values of x and y in the interval.
   (b) Use the result in part (a) to show that
   $$|\tan x - \tan y| \ge |x - y|$$
   for all values of x and y in the interval $(-\pi/2, \pi/2)$.
   (c) Use the result in part (b) to show that
   $$|\tan x + \tan y| \ge |x + y|$$
   for all values of x and y in the interval $(-\pi/2, \pi/2)$.

27. (a) Use the Mean-Value Theorem to show that
   $$\sqrt{y} - \sqrt{x} < \frac{y - x}{2\sqrt{x}}$$
   if $0 < x < y$.
   (b) Use the result in part (a) to show that if $0 < x < y$, then $\sqrt{xy} < \frac{1}{2}(x + y)$.

28. Show that if f is differentiable on an open interval and $f'(x) \ne 0$ on the interval, the equation $f(x) = 0$ can have at most one real root in the interval.

29. Use the result in Exercise 28 to show the following:
   (a) The equation $x^3 + 4x - 1 = 0$ has exactly one real root.
   (b) If $b^2 - 3ac < 0$ and if $a \ne 0$, then the equation
   $$ax^3 + bx^2 + cx + d = 0$$
   has exactly one real root.