23. The ***mechanic's rule*** for approximating square roots states that $\sqrt{a} \approx x_{n+1}$, where

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3, \ldots$$

and x_1 is any positive approximation to $\sqrt{a}$.

(a) Apply Newton's Method to

$$f(x) = x^2 - a$$

to derive the mechanic's rule.

(b) Use the mechanic's rule to approximate $\sqrt{10}$.

24. Many calculators compute reciprocals using the approximation $1/a \approx x_{n+1}$, where

$$x_{n+1} = x_n(2 - ax_n), \quad n = 1, 2, 3, \ldots$$

and x_1 is an initial approximation to $1/a$. This formula makes it possible to perform divisions using multiplications and subtractions, which is a faster procedure than dividing directly.

(a) Apply Newton's Method to

$$f(x) = \frac{1}{x} - a$$

to derive this approximation.

(b) Use the formula to approximate $\frac{1}{17}$.

25. Use Newton's Method to approximate the absolute minimum of $f(x) = \frac{1}{4}x^4 + x^2 - 5x$.

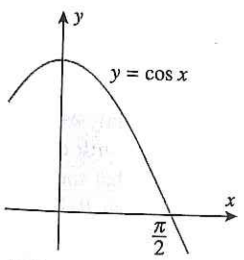
26. Use Newton's Method to approximate the absolute maximum of $f(x) = x \sin x$ on the interval $[0, \pi]$.

27. Use Newton's Method to approximate the coordinates of the the point on the parabola $y = x^2$ that is closest to the point $(1, 0)$.

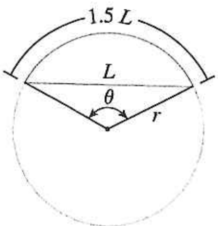
28. Use Newton's Method to approximate the dimensions of the rectangle of largest area that can be inscribed under the curve $y = \cos x$ for $0 \le x \le \pi/2$ (Figure Ex-28).

29. (a) Show that on a circle of radius r, the central angle θ that subtends an arc whose length is 1.5 times the length L of its chord satisfies the equation $\theta = 3\sin(\theta/2)$ (Figure Ex-29).

(b) Use Newton's Method to approximate θ.

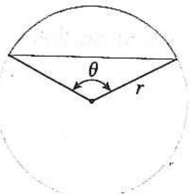


▲ Figure Ex-28



▲ Figure Ex-29

30. A ***segment*** of a circle is the region enclosed by an arc and its chord (Figure Ex-34). If r is the radius of the circle and θ the angle subtended at the center of the circle, then it can be shown that the area A of the segment is $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians. Find the value of θ for which the area of the segment is one-fourth the area of the circle. Give θ to the nearest degree.



◀ Figure Ex-30

31–32 Use Newton's Method to approximate all real values of y satisfying the given equation for the indicated value of x.

31. $xy^4 + x^3y = 1;\ x = 1$ 32. $xy - \cos\left(\frac{1}{2}xy\right) = 0;\ x = 2$

33. An ***annuity*** is a sequence of equal payments that are paid or received at regular time intervals. For example, you may want to deposit equal amounts at the end of each year into an interest-bearing account for the purpose of accumulating a lump sum at some future time. If, at the end of each year, interest of $i \times 100\%$ on the account balance for that year is added to the account, then the account is said to pay $i \times 100\%$ interest, ***compounded annually***. It can be shown that if payments of Q dollars are deposited at the end of each year into an account that pays $i \times 100\%$ compounded annually, then at the time when the nth payment and the accrued interest for the past year are deposited, the amount $S(n)$ in the account is given by the formula

$$S(n) = \frac{Q}{i}[(1 + i)^n - 1]$$

Suppose that you can invest \$5000 in an interest-bearing account at the end of each year, and your objective is to have \$250,000 on the 25th payment. Approximately what annual compound interest rate must the account pay for you to achieve your goal? [*Hint:* Show that the interest rate i satisfies the equation $50i = (1 + i)^{25} - 1$, and solve it using Newton's Method.]

FOCUS ON CONCEPTS

34. (a) Use a graphing utility to generate the graph of

$$f(x) = \frac{x}{x^2 + 1}$$

and use it to explain what happens if you apply Newton's Method with a starting value of $x_1 = 2$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

(b) Use the graph generated in part (a) to explain what happens if you apply Newton's Method with a starting value of $x_1 = 0.5$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

35. (a) Apply Newton's Method to $f(x) = x^2 + 1$ with a starting value of $x_1 = 0.5$, and determine if the values of $x_2, \ldots, x_{10}$ appear to converge.

(b) Explain what is happening.

36. In each part, explain what happens if you apply Newton's Method to a function f when the given condition is satisfied for some value of n.

(a) $f(x_n) = 0$ (b) $x_{n+1} = x_n$

(c) $x_{n+2} = x_n \neq x_{n+1}$

by a distance of 90 cm. Where on the line segment between the two sources is the total intensity a minimum?

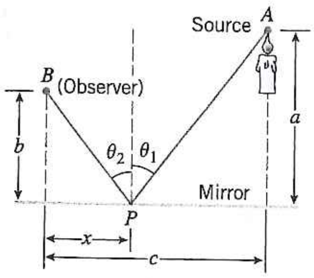
62. Given points $A(2, 1)$ and $B(5, 4)$, find the point P in the interval $[2, 5]$ on the x-axis that maximizes angle APB.

63. The lower edge of a painting, 10 ft in height, is 2 ft above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?

FOCUS ON CONCEPTS

64. ***Fermat's principle*** (biography on p. 225) in optics states that light traveling from one point to another follows that path for which the total travel time is minimum. In a uniform medium, the paths of "minimum time" and "shortest distance" turn out to be the same, so that light, if unobstructed, travels along a straight line. Assume that we have a light source, a flat mirror, and an observer in a uniform medium. If a light ray leaves the source, bounces off the mirror, and travels on to the observer, then its path will consist of two line segments, as shown in Figure Ex-64. According to Fermat's principle, the path will be such that the total travel time t is minimum or, since the medium is uniform, the path will be such that the total distance traveled from A to P to B is as small as possible. Assuming the minimum occurs when $dt/dx = 0$, show that the light ray will strike the mirror at the point P where the "angle of incidence" θ_1 equals the "angle of reflection" θ_2.

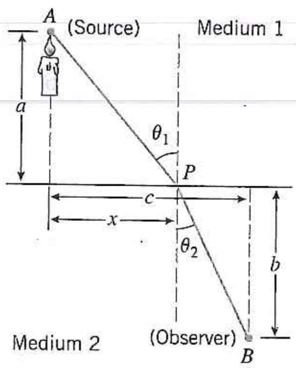
2



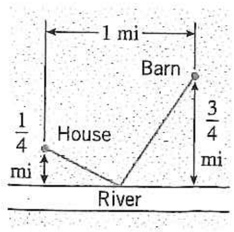
◀ **Figure Ex-64**

65. Fermat's principle (Exercise 64) also explains why light rays traveling between air and water undergo bending (refraction). Imagine that we have two uniform media (such as air and water) and a light ray traveling from a source A in one medium to an observer B in the other medium (Figure Ex-65). It is known that light travels at a constant speed in a uniform medium, but more slowly in a dense medium (such as water) than in a thin medium (such as air). Consequently, the path of shortest time from A to B is not necessarily a straight line, but rather some broken line path A to P to B allowing the light to take greatest advantage of its higher speed through the thin medium. ***Snell's law of refraction*** (biography on p. 238) states that the path of the light ray will be such that
$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$
where v_1 is the speed of light in the first medium, v_2 is the speed of light in the second medium, and θ_1 and θ_2 are the angles shown in Figure Ex-65. Show that this follows from the assumption that the path of minimum time occurs when $dt/dx = 0$.

66. A farmer wants to walk at a constant rate from her barn to a straight river, fill her pail, and carry it to her house in the least time.
(a) Explain how this problem relates to Fermat's principle and the light-reflection problem in Exercise 64.
(b) Use the result of Exercise 64 to describe geometrically the best path for the farmer to take.
(c) Use part (b) to determine where the farmer should fill her pail if her house and barn are located as in Figure Ex-66.



▲ **Figure Ex-65**



▲ **Figure Ex-66**

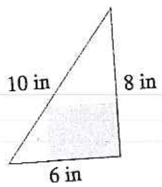
67. If an unknown physical quantity x is measured n times, the measurements $x_1, x_2, \ldots, x_n$ often vary because of uncontrollable factors such as temperature, atmospheric pressure, and so forth. Thus, a scientist is often faced with the problem of using n different observed measurements to obtain an estimate $\bar{x}$ of an unknown quantity x. One method for making such an estimate is based on the ***least squares principle***, which states that the estimate $\bar{x}$ should be chosen to minimize
$$s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2$$
which is the sum of the squares of the deviations between the estimate $\bar{x}$ and the measured values. Show that the estimate resulting from the least squares principle is
$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$
that is, $\bar{x}$ is the arithmetic average of the observed values.

68. Prove: If $f(x) \geq 0$ on an interval and if $f(x)$ has a maximum value on that interval at x_0, then $\sqrt{f(x)}$ also has a maximum value at x_0. Similarly for minimum values. [*Hint:* Use the fact that $\sqrt{x}$ is an increasing function on the interval $[0, +\infty)$.]

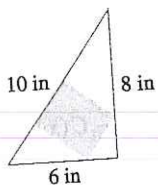
69. **Writing** Discuss the importance of finding intervals of possible values imposed by physical restrictions on variables in an applied maximum or minimum problem.

6. A rectangle is to be inscribed in a right triangle having sides of length 6 in, 8 in, and 10 in. Find the dimensions of the rectangle with greatest area assuming the rectangle is positioned as in Figure Ex-6.

7. Solve the problem in Exercise 6 assuming the rectangle is positioned as in Figure Ex-7.



▲ Figure Ex-6



▲ Figure Ex-7

8. A rectangle has its two lower corners on the x-axis and its two upper corners on the curve $y = 16 - x^2$. For all such rectangles, what are the dimensions of the one with largest area?

1

9. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10.

10. Find the point P in the first quadrant on the curve $y = x^{-2}$ such that a rectangle with sides on the coordinate axes and a vertex at P has the smallest possible perimeter.

11. A rectangular area of 3200 ft^2 is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle of least cost.

12. Show that among all rectangles with perimeter p, the square has the maximum area.

13. Show that among all rectangles with area A, the square has the minimum perimeter.

14. A wire of length 12 in can be bent into a circle, bent into a square, or cut into two pieces to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be

2

(a) a maximum (b) a minimum?

15. A rectangle R in the plane has corners at $(\pm 8, \pm 12)$, and a 100 by 100 square S is positioned in the plane so that its sides are parallel to the coordinate axes and the lower left corner of S is on the line $y = -3x$. What is the largest possible area of a region in the plane that is contained in both R and S?

16. Solve the problem in Exercise 15 if S is a 16 by 16 square.

17. Solve the problem in Exercise 15 if S is positioned with its lower left corner on the line $y = -6x$.

18. A rectangular page is to contain 42 square inches of printable area. The margins at the top and bottom of the page are each 1 inch, one side margin is 1 inch, and the other side margin is 2 inches. What should the dimensions of the page be so that the least amount of paper is used?

19. A box with a square base is taller than it is wide. In order to send the box through the U.S. mail, the height of the box and the perimeter of the base can sum to no more than 108 in. What is the maximum volume for such a box?

20. A box with a square base is wider than it is tall. In order to send the box through the U.S. mail, the width of the box and the perimeter of one of the (nonsquare) sides of the box can sum to no more than 108 in. What is the maximum volume for such a box?

21. An open box is to be made from a 3 ft by 8 ft rectangula piece of sheet metal by cutting out squares of equal siz from the four corners and bending up the sides. Find th maximum volume that the box can have.

22. A closed rectangular container with a square base is to hav a volume of 2250 in^3. The material for the top and botto of the container will cost \$2 per in^2, and the material f of the sides will cost \$3 per in^2. Find the dimensions of t container of least cost.

23. A closed rectangular container with a square base is to ha a volume of 2000 cm^3. It costs twice as much per squ centimeter for the top and bottom as it does for the sid Find the dimensions of the container of least cost.

1

24. A container with square base, vertical sides, and open to to be made from 1000 ft^2 of material. Find the dimensi of the container with greatest volume.

25. A rectangular container with two square sides and an o top is to have a volume of V cubic units. Find the di sions of the container with minimum surface area.

26. A church window consisting of a rectangle topped by a s circle is to have a perimeter p. Find the radius of the s circle if the area of the window is to be maximum.

27. Find the dimensions of the right circular cylinder of la volume that can be inscribed in a sphere of radius R.

28. Find the dimensions of the right circular cylinder of gr surface area that can be inscribed in a sphere of radiu

29. A closed, cylindrical can is to have a volume of bic units. Show that the can of minimum surface a achieved when the height is equal to the diameter base.

30. A closed cylindrical can is to have a surface area of S units. Show that the can of maximum volume is a when the height is equal to the diameter of the base.

31. A cylindrical can, open at the top, is to hold 500 liquid. Find the height and radius that minimize the of material needed to manufacture the can.

32. A soup can in the shape of a right circular cylinder o r and height h is to have a prescribed volume V. and bottom are cut from squares as shown in Figur on the next page. If the shaded corners are wasted, is no other waste, find the ratio r/h for the can requ least material (including waste).

33. A box-shaped wire frame consists of two identi squares whose vertices are connected by four strai of equal length (Figure Ex-33 on the next page

For the following functions, find where they are $\uparrow$, $\downarrow$, local max/min, concave $\uparrow$, concave $\downarrow$, points of inflection, and draw a graph

(a) $$y = \frac{(x+1)^2}{x^2+1}$$

(b) $$y = x^{2/3}\left(5/2 - x\right)$$