MA1123 Assignment5

[due Tuesday 28 October, 2014]

1. Use Mathematical Induction to prove the $(f_1 f_2 \ldots f_n)' = f_1' f_2 \ldots f_n + \ldots f_1 f_2 \ldots f_n'$. Try to be very clear in the induction process.

2. For the following functions find all asymptotes, where they increase or decrease, have local extrema, different concavities, points of inflection, etc. Then draw a rough graph.

   (a) $y = \frac{x}{(x+2)^2}$

   (b) $y = \frac{x}{x^2-4}$

3. Do 11-14, 45,49 from the book, pages 195/196 are atached

4. Do 12, 17-20 28, from pages 222 and 223, also attached.

8. Use the graph in Exercise 7 to make a table that shows the signs of f' and f'' over the intervals $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 5)$, $(5, 6)$, and $(6, 7)$.

9–10 A sign chart is presented for the first and second derivatives of a function f. Assuming that f is continuous everywhere, find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

9.

INTERVAL	SIGN OF $f'(x)$	SIGN OF $f''(x)$
$x < 1$	$-$	$+$
$1 < x < 2$	$+$	$+$
$2 < x < 3$	$+$	$-$
$3 < x < 4$	$-$	$-$
$4 < x$	$-$	$+$

10.

INTERVAL	SIGN OF $f'(x)$	SIGN OF $f''(x)$
$x < 1$	$+$	$+$
$1 < x < 3$	$+$	$-$
$3 < x$	$+$	$+$

11–14 True–False Assume that f is differentiable everywhere. Determine whether the statement is true or false. Explain your answer.

11. If f is decreasing on $[0, 2]$, then $f(0) > f(1) > f(2)$.
12. If $f'(1) > 0$, then f is increasing on $[0, 2]$.
13. If f is increasing on $[0, 2]$, then $f'(1) > 0$.
14. If f' is increasing on $[0, 1]$ and f' is decreasing on $[1, 2]$, then f has an inflection point at $x = 1$.

15–26 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

15. $f(x) = x^2 - 3x + 8$
16. $f(x) = 5 - 4x - x^2$
17. $f(x) = (2x + 1)^3$
18. $f(x) = 5 + 12x - x^3$
19. $f(x) = 3x^4 - 4x^3$
20. $f(x) = x^4 - 5x^3 + 9x^2$
21. $f(x) = \dfrac{x - 2}{(x^2 - x + 1)^2}$
22. $f(x) = \dfrac{x}{x^2 + 2}$
23. $f(x) = \sqrt[3]{x^2 + x + 1}$
24. $f(x) = x^{4/3} - x^{1/3}$
25. $f(x) = (x^{2/3} - 1)^2$
26. $f(x) = x^{2/3} - x$

27–32 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility.

27. $f(x) = \sin x - \cos x;\ [-\pi, \pi]$
28. $f(x) = \sec x \tan x;\ (-\pi/2, \pi/2)$
29. $f(x) = 1 - \tan(x/2);\ (-\pi, \pi)$
30. $f(x) = 2x + \cot x;\ (0, \pi)$
31. $f(x) = (\sin x + \cos x)^2;\ [-\pi, \pi]$
32. $f(x) = \sin^2 2x;\ [0, \pi]$

FOCUS ON CONCEPTS

33. In parts (a)–(c), sketch a continuous curve $y = f(x)$ with the stated properties.
    (a) $f(2) = 4,\ f'(2) = 0,\ f''(x) > 0$ for all x
    (b) $f(2) = 4,\ f'(2) = 0,\ f''(x) < 0$ for $x < 2,\ f''(x) > 0$ for $x > 2$
    (c) $f(2) = 4,\ f''(x) < 0$ for $x \neq 2$ and $\lim_{x\to 2^+} f'(x) = +\infty,\ \lim_{x\to 2^-} f'(x) = -\infty$
34. In each part sketch a continuous curve $y = f(x)$ with the stated properties.
    (a) $f(2) = 4,\ f'(2) = 0,\ f''(x) < 0$ for all x
    (b) $f(2) = 4,\ f'(2) = 0,\ f''(x) > 0$ for $x < 2,\ f''(x) < 0$ for $x > 2$
    (c) $f(2) = 4,\ f''(x) > 0$ for $x \neq 2$ and $\lim_{x\to 2^+} f'(x) = -\infty,\ \lim_{x\to 2^-} f'(x) = +\infty$

35–38 If f is increasing on an interval $[0, b)$, then it follows from Definition 3.1.1 that $f(0) < f(x)$ for each x in the interval $(0, b)$. Use this result in these exercises.

35. Show that $\sqrt[3]{1 + x} < 1 + \frac{1}{3}x$ if $x > 0$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = 1 + \frac{1}{3}x - \sqrt[3]{1 + x}$ is increasing on $[0, +\infty)$.]
36. Show that $x < \tan x$ if $0 < x < \pi/2$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = \tan x - x$ is increasing on $[0, \pi/2)$.]
37. Use a graphing utility to make a conjecture about the relative sizes of x and $\sin x$ for $x \geq 0$, and prove your conjecture.
38. Use a graphing utility to make a conjecture about the relative sizes of $1 - x^2/2$ and $\cos x$ for $x \geq 0$, and prove your conjecture. [*Hint:* Use the result of Exercise 37.]

39–40 Use a graphing utility to generate the graphs of f' and f'' over the stated interval; then use those graphs to estimate the x-coordinates of the inflection points of f, the intervals on which f is concave up or down, and the intervals on which f is increasing or decreasing. Check your estimates by graphing f.

39. $f(x) = x^4 - 24x^2 + 12x,\quad -5 \leq x \leq 5$
40. $f(x) = \dfrac{1}{1 + x^2},\quad -5 \leq x \leq 5$

C **41–42** Use a CAS to find f'' and to approximate the x-coordinates of the inflection points to six decimal places. Confirm that your answer is consistent with the graph of f.

41. $f(x) = \dfrac{10x - 3}{3x^2 - 5x + 8}$

42. $f(x) = \dfrac{x^3 - 8x + 7}{\sqrt{x^2 + 1}}$

43. Use Definition 3.1.1 to prove that $f(x) = x^2$ is increasing on $[0, +\infty)$.

44. Use Definition 3.1.1 to prove that $f(x) = 1/x$ is decreasing on $(0, +\infty)$.

FOCUS ON CONCEPTS

45–48 Determine whether the statements are true or false. If a statement is false, find functions for which the statement fails to hold.

45. (a) If f and g are increasing on an interval, then so is $f + g$.
(b) If f and g are increasing on an interval, then so is $f \cdot g$.

46. (a) If f and g are concave up on an interval, then so is $f + g$.
(b) If f and g are concave up on an interval, then so is $f \cdot g$.

47. In each part, find functions f and g that are increasing on $(-\infty, +\infty)$ and for which $f - g$ has the stated property.
(a) $f - g$ is decreasing on $(-\infty, +\infty)$.
(b) $f - g$ is constant on $(-\infty, +\infty)$.
(c) $f - g$ is increasing on $(-\infty, +\infty)$.

48. In each part, find functions f and g that are positive and increasing on $(-\infty, +\infty)$ and for which f/g has the stated property.
(a) f/g is decreasing on $(-\infty, +\infty)$.
(b) f/g is constant on $(-\infty, +\infty)$.
(c) f/g is increasing on $(-\infty, +\infty)$.

49. (a) Prove that a general cubic polynomial
$$f(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$
has exactly one inflection point.
(b) Prove that if a cubic polynomial has three x-intercepts, then the inflection point occurs at the average value of the intercepts.
(c) Use the result in part (b) to find the inflection point of the cubic polynomial $f(x) = x^3 - 3x^2 + 2x$, and check your result by using f'' to determine where f is concave up and concave down.

50. From Exercise 49, the polynomial $f(x) = x^3 + bx^2 + 1$ has one inflection point. Use a graphing utility to reach a conclusion about the effect of the constant b on the location of the inflection point. Use f'' to explain what you have observed graphically.

51. Use Definition 3.1.1 to prove:
(a) If f is increasing on the intervals $(a, c]$ and $[c, b)$, then f is increasing on (a, b).
(b) If f is decreasing on the intervals $(a, c]$ and $[c, b)$, then f is decreasing on (a, b).

52. Use part (a) of Exercise 51 to show that $f(x) = x + \sin x$ is increasing on the interval $(-\infty, +\infty)$.

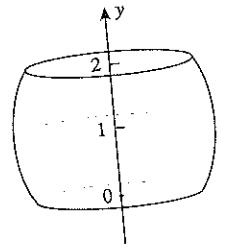
53. Use part (b) of Exercise 51 to show that $f(x) = \cos x - x$ is decreasing on the interval $(-\infty, +\infty)$.

54. Let $y = 1/(1 + x^2)$. Find the values of x for which y is increasing most rapidly or decreasing most rapidly.

FOCUS ON CONCEPTS

55–58 Suppose that water is flowing at a constant rate into the container shown. Make a rough sketch of the graph of the water level y versus the time t. Make sure that your sketch conveys where the graph is concave up and concave down, and label the y-coordinates of the inflection points.

55.



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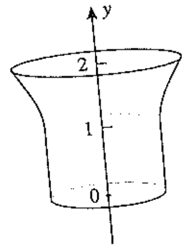

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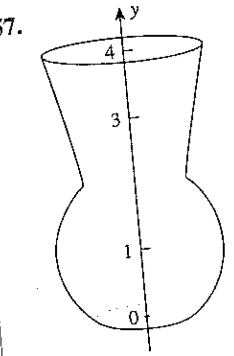

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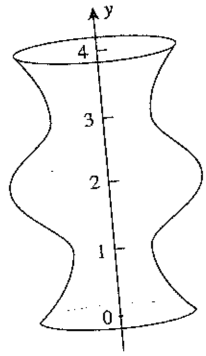


59. **Writing** An approaching storm causes the air temperature to fall. Make a statement that indicates there is an inflection point in the graph of temperature versus time. Explain how the existence of an inflection point follows from your statement.

60. **Writing** Explain what the sign analyses of $f'(x)$ and $f''(x)$ tell us about the graph of $y = f(x)$.

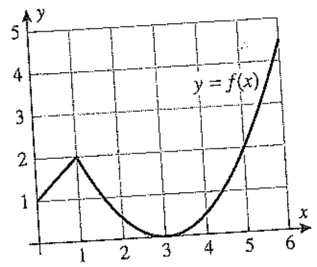
✔ QUICK CHECK ANSWERS 3.1

1. (a) $f(x_1) < f(x_2)$ (b) $f(x_1) > f(x_2)$ (c) increasing (d) $= 0$ 2. (a) $-1, 3$ (b) $(-\infty, -1]$ and $[3, +\infty)$ (c) $(-\infty, 1)$ (d) $(1, -1.1)$ 3. (a) $[0, +\infty)$ (b) $\left(-\infty, \frac{4}{3}\right), (4, +\infty)$ (c) $\left(\frac{4}{3}, 4\right)$ 4. The graph is increasing and concave down.

✔ QUICK CHECK EXERCISES 3.4

(See page 224 for answers.)

1. Use the accompanying graph to find the x-coordinates of the relative extrema and absolute extrema of f on $[0, 6]$.



◀ Figure Ex-1

2. Suppose that a function f is continuous on $[-4, 4]$ and has critical points at $x = -3, 0, 2$. Use the accompanying table to determine the absolute maximum and absolute minimum values, if any, for f on the indicated intervals.
(a) $[1, 4]$ (b) $[-2, 2]$ (c) $[-4, 4]$ (d) $(-4, 4)$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	2224	-1333	0	1603	2096	2293	2400	2717	6064

3. Let $f(x) = x^3 - 3x^2 - 9x + 25$. Use the derivative $f'(x) = 3(x+1)(x-3)$ to determine the absolute maximum and absolute minimum values, if any, for f on each of the given intervals.
(a) $[0, 4]$ (b) $[-2, 4]$ (c) $[-4, 2]$
(d) $[-5, 10]$ (e) $(-5, 4)$

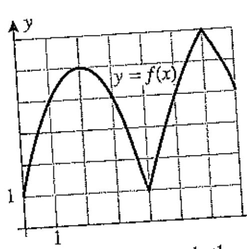
EXERCISE SET 3.4

Graphing Utility C CAS

FOCUS ON CONCEPTS

1–2 Use the graph to find x-coordinates of the relative extrema and absolute extrema of f on $[0, 7]$.

1.



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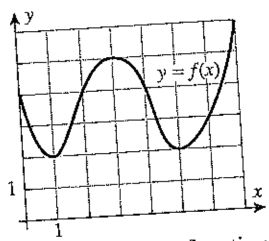

3. In each part, sketch the graph of a continuous function f with the stated properties on the interval $[0, 10]$.
(a) f has an absolute minimum at $x = 0$ and an absolute maximum at $x = 10$.
(b) f has an absolute minimum at $x = 2$ and an absolute maximum at $x = 7$.
(c) f has relative minima at $x = 1$ and $x = 8$, has relative maxima at $x = 3$ and $x = 7$, has an absolute minimum at $x = 5$, and has an absolute maximum at $x = 10$.

4. In each part, sketch the graph of a continuous function f with the stated properties on the interval $(-\infty, +\infty)$.
(a) f has no relative extrema or absolute extrema.
(b) f has an absolute minimum at $x = 0$ but no absolute maximum.
(c) f has an absolute maximum at $x = -5$ and an absolute minimum at $x = 5$.

5. Let
$$f(x) = \begin{cases} \dfrac{1}{1-x}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$
Explain why f has a minimum value but no maximum value on the closed interval $[0, 1]$.

6. Let
$$f(x) = \begin{cases} x, & 0 < x < 1 \\ \frac{1}{2}, & x = 0, 1 \end{cases}$$
Explain why f has neither a minimum value nor a maximum value on the closed interval $[0, 1]$.

7–16 Find the absolute maximum and minimum values of f on the given closed interval, and state where those values occur.

7. $f(x) = 4x^2 - 12x + 10;\ [1, 2]$
8. $f(x) = 8x - x^2;\ [0, 6]$
9. $f(x) = (x-2)^3;\ [1, 4]$
10. $f(x) = 2x^3 + 3x^2 - 12x;\ [-3, 2]$
11. $f(x) = \dfrac{3x}{\sqrt{4x^2+1}};\ [-1, 1]$
12. $f(x) = (x^2 + x)^{2/3};\ [-2, 3]$
13. $f(x) = x - 2\sin x;\ [-\pi/4, \pi/2]$
14. $f(x) = \sin x - \cos x;\ [0, \pi]$
15. $f(x) = 1 + |9 - x^2|;\ [-5, 1]$
16. $f(x) = |6 - 4x|;\ [-3, 3]$

17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. If a function f is continuous on $[a, b]$, then f has an absolute maximum on $[a, b]$.
18. If a function f is continuous on (a, b), then f has an absolute minimum on (a, b).
19. If a function f has an absolute minimum on (a, b), then there is a critical point of f in (a, b).
20. If a function f is continuous on $[a, b]$ and f has no relative extreme values in (a, b), then the absolute maximum value of f exists and occurs either at $x = a$ or at $x = b$.

21–28 Find the absolute maximum and minimum values of f, if any, on the given interval, and state where those values occur.

21. $f(x) = x^2 - x - 2;\ (-\infty, +\infty)$
22. $f(x) = 3 - 4x - 2x^2;\ (-\infty, +\infty)$
23. $f(x) = 4x^3 - 3x^4;\ (-\infty, +\infty)$
24. $f(x) = x^4 + 4x;\ (-\infty, +\infty)$
25. $f(x) = 2x^3 - 6x + 2;\ (-\infty, +\infty)$
26. $f(x) = x^3 - 9x + 1;\ (-\infty, +\infty)$
27. $f(x) = \dfrac{x^2 + 1}{x + 1};\ (-5, -1)$
28. $f(x) = \dfrac{x - 2}{x + 1};\ (-1, 5]$

29–38 Use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values.

29. $f(x) = (x^2 - 2x)^2;\ (-\infty, +\infty)$
30. $f(x) = (x - 1)^2(x + 2)^2;\ (-\infty, +\infty)$
31. $f(x) = x^{2/3}(20 - x);\ [-1, 20]$
32. $f(x) = \dfrac{x}{x^2 + 2};\ [-1, 4]$
33. $f(x) = 1 + \dfrac{1}{x};\ (0, +\infty)$
34. $f(x) = \dfrac{2x^2 - 3x + 3}{x^2 - 2x + 2};\ [1, +\infty)$
35. $f(x) = \dfrac{2 - \cos x}{\sin x};\ [\pi/4, 3\pi/4]$
36. $f(x) = \sin^2 x + \cos x;\ [-\pi, \pi]$
37. $f(x) = \sin(\cos x);\ [0, 2\pi]$
38. $f(x) = \cos(\sin x);\ [0, \pi]$
39. Find the absolute maximum and minimum values of
$$f(x) = \begin{cases} 4x - 2, & x < 1 \\ (x - 2)(x - 3), & x \geq 1 \end{cases}$$
on $\left[\frac{1}{2}, \frac{7}{2}\right]$.
40. Let $f(x) = x^2 + px + q$. Find the values of p and q such that $f(1) = 3$ is an extreme value of f on $[0, 2]$. Is this value a maximum or minimum?

41–42 If f is a periodic function, then the locations of all absolute extrema on the interval $(-\infty, +\infty)$ can be obtained by finding the locations of the absolute extrema for one period and using the periodicity to locate the rest. Use this idea in these exercises to find the absolute maximum and minimum values of the function, and state the x-values at which they occur.

41. $f(x) = 2\cos x + \cos 2x$
42. $f(x) = 3\cos\dfrac{x}{3} + 2\cos\dfrac{x}{2}$

43–44 One way of proving that $f(x) \leq g(x)$ for all x in a given interval is to show that $0 \leq g(x) - f(x)$ for all x in the interval; and one way of proving the latter inequality is to show that the absolute minimum value of $g(x) - f(x)$ on the interval is nonnegative. Use this idea to prove the inequalities in these exercises.

43. Prove that $\sin x \leq x$ for all x in the interval $[0, 2\pi]$.
44. Prove that $\cos x \geq 1 - (x^2/2)$ for all x in the interval $[0, 2\pi]$.
45. What is the smallest possible slope for a tangent to the graph of the equation $y = x^3 - 3x^2 + 5x$?
46. (a) Show that $f(x) = \sec x + \csc x$ has a minimum value but no maximum value on the interval $(0, \pi/2)$.
(b) Find the minimum value in part (a).
47. C Show that the absolute minimum value of
$$f(x) = x^2 + \frac{x^2}{(8 - x)^2}, \quad x > 8$$
occurs at $x = 10$ by using a CAS to find $f'(x)$ and to solve the equation $f'(x) = 0$.
48. C The vertical displacement $f(t)$ of a cork bobbing up and down on the ocean's surface may be modeled by the function
$$f(t) = A\cos t + B\sin t$$
where $A > 0$ and $B > 0$. Use a CAS to find the maximum and minimum values of $f(t)$ in terms of A and B.
49. Suppose that the equations of motion of a paper airplane during the first 12 seconds of flight are
$$x = t - 2\sin t, \quad y = 2 - 2\cos t \quad (0 \leq t \leq 12)$$
What are the highest and lowest points in the trajectory, and when is the airplane at those points?
50. The accompanying figure shows the path of a fly whose equations of motion are
$$x = \frac{\cos t}{2 + \sin t}, \quad y = 3 + \sin(2t) - 2\sin^2 t \quad (0 \leq t \leq 2\pi)$$
(a) How high and low does it fly?
(b) How far left and right of the origin does it fly?

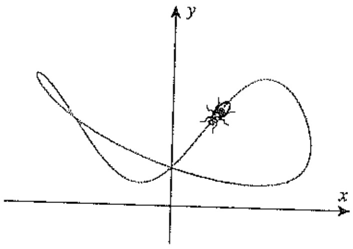

Figure Ex-50

51. Let $f(x) = ax^2 + bx + c$, where $a > 0$. Prove that $f(x) \geq 0$ for all x if and only if $b^2 - 4ac \leq 0$. [*Hint:* Find the minimum of $f(x)$.]
52. Prove Theorem 3.4.3 in the case where the extreme value is a minimum.
53. **Writing** Suppose that f is continuous and positive-valued everywhere and that the x-axis is an asymptote for the graph