MA1123 Assignment10

[due Monday 25th January2016]

1. Solve $(D+2)^3y = 0$ in three steps. Note we did $(D+a)^2y = 0$ in two steps.
2. Solve $(D+2)^3y = 3x$
3. Solve $(D+2)(D+3)y = 2\cos 3x$

Now do the questions that are marked on the attached sheets.

The initial-value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(0) = 1$$

has solution $y(x) =$ ______.

(ERCISE SET 8.2 ☐ Graphing Utility ☐C CAS

-10 Solve the differential equation by separation of variables. here reasonable, express the family of solutions as explicit nctions of x.

1. $\frac{dy}{dx} = \frac{y}{x}$
2. $\frac{dy}{dx} = 4(1+y^2)x$
3. $\frac{\sqrt{1+x^2}}{1+y}\frac{dy}{dx} = -x$
4. $(1+x^4)\frac{dy}{dx} = \frac{x^3}{y}$
5. $(2+2y^2)y' = e^x y$
6. $y' = -x^2 y$
7. $e^{-y}\sin x - y'\cos^2 x = 0$
8. $y' - (1+x)(1+y^2) = 0$
9. $\frac{dy}{dx} - \frac{y^2 - y}{\sin x} = 0$
10. $y - \frac{dy}{dx}\csc x = 0$

1–14 Solve the initial-value problem by separation of variables.

1. $y' = \frac{3x^2}{2y + \cos y}, \quad y(0) = \pi$
2. $y' - xe^y = 2e^y, \quad y(0) = 0$
13. $\frac{dy}{dt} = \frac{2t+1}{2y-2}, \quad y(0) = 1$
14. $y'\cosh^2 x - y\cosh 2x = 0, \quad y(0) = 5$

15. (a) Sketch some typical integral curves of the differential equation $y' = y/2x$.
(b) Find an equation for the integral curve that passes through the point $(3, 1)$.

16. (a) Sketch some typical integral curves of the differential equation $y' = -x/y$.
(b) Find an equation for the integral curve that passes through the point $(3, 4)$.

17–18 Solve the differential equation and then use a graphing utility to generate five integral curves for the equation.

17. $(x^2+4)\frac{dy}{dx} + xy = 0$
18. $(\cos y)y' = \cos x$

19–20 Solve the differential equation. If you have a CAS with implicit plotting capability, use the CAS to generate five integral curves for the equation.

19. $y' = \frac{x^2}{1-y^2}$
20. $y' = \frac{y}{1+y^2}$

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. Every differential equation of the form $y' = f(y)$ is separable.

22. If a population is growing exponentially, then the time it takes the population to quadruple is independent of the size of the population.

23. If a radioactive element has a half-life of 1 minute, and if a container holds 32 g of the element at 1:00 P.M., then the amount remaining at 1:05 P.M. will be 1 g.

24. A differential equation of the form

$$h(x)\frac{dy}{dx} = g(y)$$

is not separable.

25. Suppose that the initial condition in Example 1 had been $y(0) = 0$. Show that none of the solutions generated in Example 1 satisfy this initial condition, and then solve the initial-value problem

$$\frac{dy}{dx} = -4xy^2, \quad y(0) = 0$$

Why does the method of Example 1 fail to produce this particular solution?

26. Find all ordered pairs (x_0, y_0) such that if the initial condition in Example 1 is replaced by $y(x_0) = y_0$, the solution of the resulting initial-value problem is defined for all real numbers.

27. Find an equation of a curve with x-intercept 4 whose tangent line at any point (x, y) has slope xe^{-y}.

28. Use a graphing utility to generate a curve that passes through the point $(1, 1)$ and whose tangent line at (x, y) is perpendicular to the line through (x, y) with slope $-2y/(3x^2)$.

29. Suppose that an initial population of 20,000 bacteria grows exponentially at a rate of 2% per hour and that $y = y(t)$ is the number of bacteria present t hours later.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.
(c) How long does it take for the initial population of bacteria to double?
(d) How long does it take for the population of bacteria to reach 45,000?

30. A cell of the bacterium *E. coli* divides into two cells every 20 minutes when placed in a nutrient culture. Let $y = y(t)$ be the number of cells that are present t minutes after a single cell is placed in the culture. Assume that the growth of the bacteria is approximated by an exponential growth model.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$. (cont.)

(c) How many cells are present after 2 hours?
(d) How long does it take for the number of cells to reach 1,000,000?

31. Radon-222 is a radioactive gas with a half-life of 3.83 days. This gas is a health hazard because it tends to get trapped in the basements of houses, and many health officials suggest that homeowners seal their basements to prevent entry of the gas. Assume that 5.0×10^7 radon atoms are trapped in a basement at the time it is sealed and that $y(t)$ is the number of atoms present t days later.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.
(c) How many atoms will be present after 30 days?
(d) How long will it take for 90% of the original quantity of gas to decay?

32. Polonium-210 is a radioactive element with a half-life of 140 days. Assume that 20 milligrams of the element are placed in a lead container and that $y(t)$ is the number of milligrams present t days later.
(a) Find an initial-value problem whose solution is $y(t)$.
(b) Find a formula for $y(t)$.
(c) How many milligrams will be present after 10 weeks?
(d) How long will it take for 70% of the original sample to decay?

33. Suppose that 200 fruit flies are placed in a breeding container that can support at most 10,000 flies. Assuming that the population grows exponentially at a rate of 2% per day, how long will it take for the container to reach capacity?

34. Suppose that the town of Grayrock had a population of 10,000 in 2006 and a population of 12,000 in 2011. Assuming an exponential growth model, in what year will the population reach 20,000?

35. A scientist wants to determine the half-life of a certain radioactive substance. She determines that in exactly 5 days a 10.0-milligram sample of the substance decays to 3.5 milligrams. Based on these data, what is the half-life?

36. Suppose that 40% of a certain radioactive substance decays in 5 years.
(a) What is the half-life of the substance in years?
(b) Suppose that a certain quantity of this substance is stored in a cave. What percentage of it will remain after t years?

FOCUS ON CONCEPTS

37. (a) Make a conjecture about the effect on the graphs of $y = y_0e^{kt}$ and $y = y_0e^{-kt}$ of varying k and keeping y_0 fixed. Confirm your conjecture with a graphing utility.
(b) Make a conjecture about the effect on the graphs of $y = y_0e^{kt}$ and $y = y_0e^{-kt}$ of varying y_0 and keeping k fixed. Confirm your conjecture with a graphing utility.

38. (a) What effect does increasing y_0 and keeping k fixed have on the doubling time or half-life of an exponential model? Justify your answer.
(b) What effect does increasing k and keeping y_0 fixed have on the doubling time and half-life of an exponential model? Justify your answer.

39. (a) There is a trick, called the ***Rule of 70***, that can be used to get a quick estimate of the doubling time or half-life of an exponential model. According to this rule, the doubling time or half-life is roughly 70 divided by the percentage growth or decay rate. For example, we showed in Example 5 that with a continued growth rate of 1.10% per year the world population would double every 63 years. This result agrees with the Rule of 70, since $70/1.10 \approx 63.6$. Explain why this rule works.
(b) Use the Rule of 70 to estimate the doubling time of a population that grows exponentially at a rate of 2% per year.
(c) Use the Rule of 70 to estimate the half-life of a population that decreases exponentially at a rate of 3.5% per hour.
(d) Use the Rule of 70 to estimate the growth rate that would be required for a population growing exponentially to double every 10 years.

40. Find a formula for the tripling time of an exponential growth model.

41. In 1950, a research team digging near Folsom, New Mexico, found charred bison bones along with some leaf-shaped projectile points (called the "Folsom points") that had been made by a Paleo-Indian hunting culture. It was clear from the evidence that the bison had been cooked and eaten by the makers of the points, so that carbon-14 dating of the bones made it possible for the researchers to determine when the hunters roamed North America. Tests showed that the bones contained between 27% and 30% of their original carbon-14. Use this information to show that the hunters lived roughly between 9000 B.C. and 8000 B.C.

42. (a) Use a graphing utility to make a graph of p_{rem} versus t, where p_{rem} is the percentage of carbon-14 that remains in an artifact after t years.
(b) Use the graph to estimate the percentage of carbon-14 that would have to have been present in the 1988 test of the Shroud of Turin for it to have been the burial shroud of Jesus of Nazareth (see Example 7).

43. (a) It is currently accepted that the half-life of carbon-14 might vary ± 40 years from its nominal value of 5730 years. Does this variation make it possible that the Shroud of Turin dates to the time of Jesus of Nazareth (see Example 7)?
(b) Review the subsection of Section 2.9 entitled Error Propagation, and then estimate the percentage error that

results in the computed age of an artifact from an $r\%$ error in the half-life of carbon-14.

44. Suppose that a quantity y has an exponential growth model $y = y_0e^{kt}$ or an exponential decay model $y = y_0e^{-kt}$, and it is known that $y = y_1$ if $t = t_1$. In each case find a formula for k in terms of y_0, y_1, and t_1, assuming that $t_1 \neq 0$.

45. (a) Show that if a quantity $y = y(t)$ has an exponential model, and if $y(t_1) = y_1$ and $y(t_2) = y_2$, then the doubling time or the half-life T is

$$T = \left|\frac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)}\right|$$

(b) In a certain 1-hour period the number of bacteria in a colony increases by 30%. Assuming an exponential growth model, what is the doubling time for the colony?

46. Suppose that P dollars is invested at an annual interest rate of $r \times 100\%$. If the accumulated interest is credited to the account at the end of the year, then the interest is said to be *compounded annually*; if it is credited at the end of each 6-month period, then it is said to be *compounded semiannually*; and if it is credited at the end of each 3-month period, then it is said to be *compounded quarterly*. The more frequently the interest is compounded, the better it is for the investor since more of the interest is itself earning interest.

(a) Show that if interest is compounded n times a year at equally spaced intervals, then the value A of the investment after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

(b) One can imagine interest to be compounded each day, each hour, each minute, and so forth. Carried to the limit one can conceive of interest compounded at each instant of time; this is called ***continuous compounding***. Thus, from part (a), the value A of P dollars after t years when invested at an annual rate of $r \times 100\%$, compounded continuously, is

$$A = \lim_{n\to+\infty} P\left(1 + \frac{r}{n}\right)^{nt}$$

Use the fact that $\lim_{x\to 0}(1 + x)^{1/x} = e$ to prove that $A = Pe^{rt}$.

(c) Use the result in part (b) to show that money invested at continuous compound interest increases at a rate proportional to the amount present.

47. (a) If \$1000 is invested at 7% per year compounded continuously (Exercise 46), what will the investment be worth after 5 years?

(b) If it is desired that an investment at 7% per year compounded continuously should have a value of \$10,000 after 10 years, how much should be invested now?

(c) How long does it take for an investment at 7% per year compounded continuously to double in value?

48. What is the effective annual interest rate for an interest rate of $r\%$ per year compounded continuously?

49. Assume that $y = y(t)$ satisfies the logistic equation with $y_0 = y(0)$ the initial value of y.

(a) Use separation of variables to derive the solution

$$y = \frac{y_0L}{y_0 + (L - y_0)e^{-kt}}$$

(b) Use part (a) to show that $\lim_{t\to+\infty} y(t) = L$.

50. Use your answer to Exercise 49 to derive a solution to the model for the spread of disease [Equation (6) of Section 8.1].

51. The graph of a solution to the logistic equation is known as a ***logistic curve***, and if $y_0 > 0$, it has one of four general shapes, depending on the relationship between y_0 and L. In each part, assume that $k = 1$ and use a graphing utility to plot a logistic curve satisfying the given condition.

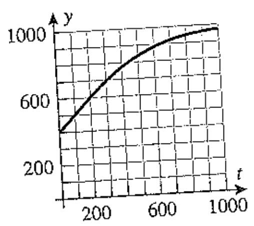
(a) $y_0 > L$
(b) $y_0 = L$
(c) $L/2 \le y_0 < L$
(d) $0 < y_0 < L/2$

52–53 The graph of a logistic model

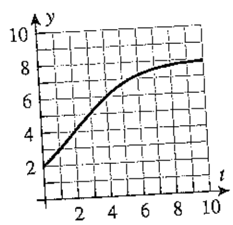
$$y = \frac{y_0L}{y_0 + (L - y_0)e^{-kt}}$$

is shown. Estimate y_0, L, and k.

52.



53.



54. Plot a solution to the initial-value problem

$$\frac{dy}{dt} = 0.98\left(1 - \frac{y}{5}\right)y, \quad y_0 = 1$$

55. Suppose that the growth of a population $y = y(t)$ is given by the logistic equation

$$y = \frac{60}{5 + 7e^{-t}}$$

(a) What is the population at time $t = 0$?
(b) What is the carrying capacity L?
(c) What is the constant k?
(d) When does the population reach half of the carrying capacity?
(e) Find an initial-value problem whose solution is $y(t)$.

56. Suppose that the growth of a population $y = y(t)$ is given by the logistic equation

$$y = \frac{500}{1 + 499e^{-0.9t}}$$

(a) What is the population at time $t = 0$?
(b) What is the carrying capacity L?
(c) What is the constant k?

(cont.)

QUICK CHECK EXERCISES 8.4 (See page 594 for answers.)

1. Solve the first-order linear differential equation

$$\frac{dy}{dx} + p(x)y = q(x)$$

by completing the following steps:

Step 1. Calculate the integrating factor $\mu =$ ______.

Step 2. Multiply both sides of the equation by the integrating factor and express the result as

$$\frac{d}{dx}[\_\_\_\_\_\_] = \_\_\_\_\_\_$$

Step 3. Integrate both sides of the equation obtained in Step 2 and solve for $y =$ ______.

2. An integrating factor for

$$\frac{dy}{dx} + \frac{y}{x} = q(x)$$

is ______.

3. At time $t = 0$, a tank contains 30 oz of salt dissolved in 60 gal of water. Then brine containing 5 oz of salt per gallon of brine is allowed to enter the tank at a rate of 3 gal/min and the mixed solution is drained from the tank at the same rate. Give an initial-value problem satisfied by the amount of salt $y(t)$ in the tank at time t. Do not solve the problem.

EXERCISE SET 8.4 Graphing Utility

1–6 Solve the differential equation by the method of integrating factors.

1. $\dfrac{dy}{dx} + 5y = e^{-3x}$

2. $\dfrac{dy}{dx} + 2xy = 3x$

3. $y' + y = \sin(e^x)$

4. $2\dfrac{dy}{dx} + 4y = 3$

5. $(x^2 + 1)\dfrac{dy}{dx} + xy = 0$

6. $\dfrac{dy}{dx} + y + \dfrac{1}{1 - e^x} = 0$

7–10 Solve the initial-value problem.

7. $x\dfrac{dy}{dx} + y = x, \quad y(1) = 3$

8. $x\dfrac{dy}{dx} - y = x^2, \quad y(1) = 1$

9. $\dfrac{dy}{dx} - 2xy = 4x, \quad y(0) = 3$

10. $\dfrac{dy}{dt} + y = 2, \quad y(0) = 3$

11–14 True–False Determine whether the statement is true or false. Explain your answer.

11. In a mixing problem, we expect the concentration of the dissolved substance within the tank to approach a finite limit over time.

12. If the first-order linear differential equation

$$\frac{dy}{dx} + p(x)y = q(x)$$

has a solution that is a constant function, then $q(x)$ is a constant multiple of $p(x)$.

13. If y_1 and y_2 are two solutions to a first-order linear differential equation, then $y = y_1 + y_2$ is also a solution.

14. In our model for free-fall motion retarded by air resistance, the terminal velocity is proportional to the weight of the falling object.

15. A slope field for the differential equation $y' = 2y - x$ is shown in the accompanying figure. In each part, sketch the graph of the solution that satisfies the initial condition.
(a) $y(1) = 1$ (b) $y(0) = -1$ (c) $y(-1) = 0$

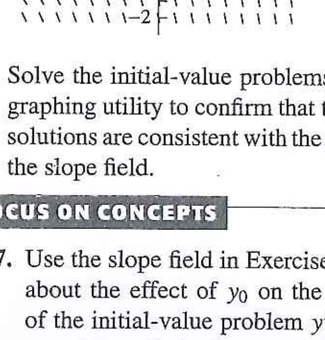


Figure Ex-15

16. Solve the initial-value problems in Exercise 15, and use a graphing utility to confirm that the integral curves for these solutions are consistent with the sketches you obtained from the slope field.

FOCUS ON CONCEPTS

17. Use the slope field in Exercise 15 to make a conjecture about the effect of y_0 on the behavior of the solution of the initial-value problem $y' = 2y - x$, $y(0) = y_0$ as $x \to +\infty$, and check your conjecture by examining the solution of the initial-value problem.

18. Consider the slope field in Exercise 15.
(a) Use Euler's Method with $\Delta x = 0.1$ to estimate $y(\frac{1}{2})$ for the solution that satisfies the initial condition $y(0) = 2$.

Sometimes new functions actually originate as power series, and the properties of the functions are developed by working with their power series representations. For example, the functions

$$J_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}=1-\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+\cdots \qquad (4)$$

and

$$J_1(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2^{2k+1}(k!)(k+1)!}=\frac{x}{2}-\frac{x^3}{2^3(1!)(2!)}+\frac{x^5}{2^5(2!)(3!)}-\cdots \qquad (5)$$

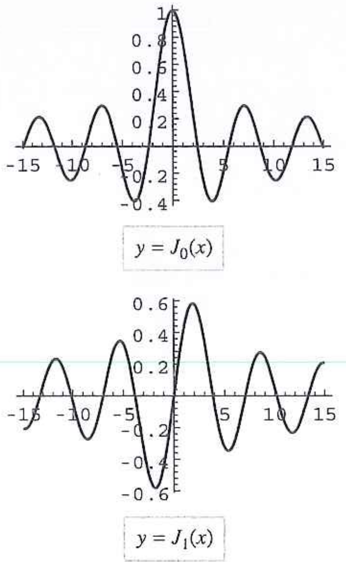
which are called ***Bessel functions*** in honor of the German mathematician and astronomer Friedrich Wilhelm Bessel (1784–1846), arise naturally in the study of planetary motion and in various problems that involve heat flow.

To find the domains of these functions, we must determine where their defining power series converge. For example, in the case of $J_0(x)$ we have

$$\rho=\lim_{k\to+\infty}\left|\frac{u_{k+1}}{u_k}\right|=\lim_{k\to+\infty}\left|\frac{x^{2(k+1)}}{2^{2(k+1)}[(k+1)!]^2}\cdot\frac{2^{2k}(k!)^2}{x^{2k}}\right|$$

$$=\lim_{k\to+\infty}\left|\frac{x^2}{4(k+1)^2}\right|=0<1$$

so the series converges for all x; that is, the domain of $J_0(x)$ is $(-\infty,+\infty)$. We leave it as an exercise (Exercise 59) to show that the power series for $J_1(x)$ also converges for all x. Computer-generated graphs of $J_0(x)$ and $J_1(x)$ are shown in Figure 9.8.4.



Generated by Mathematica

▲ **Figure 9.8.4**

TECHNOLOGY MASTERY

Many computer algebra systems have the Bessel functions as part of their libraries. If you have a CAS with Bessel functions, use it to generate the graphs in Figure 9.8.4.

✔ QUICK CHECK EXERCISES 9.8 *(See page 668 for answers.)*

1. If f has derivatives of all orders at x_0, then the Taylor series for f about $x=x_0$ is defined to be

$$\sum_{k=0}^{\infty}\_\_\_\_\_\_\_\_$$

2. Since

$$\lim_{k\to+\infty}\left|\frac{2^{k+1}x^{k+1}}{2^k x^k}\right|=2|x|$$

the radius of convergence for the infinite series $\sum_{k=0}^{\infty}2^k x^k$ is ________.

3. Since

$$\lim_{k\to+\infty}\left|\frac{(3^{k+1}x^{k+1})/(k+1)!}{(3^k x^k)/k!}\right|=\lim_{k\to+\infty}\left|\frac{3x}{k+1}\right|=0$$

the interval of convergence for the series $\sum_{k=0}^{\infty}(3^k/k!)x^k$ is ________.

4. (a) Since

$$\lim_{k\to+\infty}\left|\frac{(x-4)^{k+1}/\sqrt{k+1}}{(x-4)^k/\sqrt{k}}\right|=\lim_{k\to+\infty}\left|\sqrt{\frac{k}{k+1}}(x-4)\right|$$

$$=|x-4|$$

the radius of convergence for the infinite series $\sum_{k=1}^{\infty}(1/\sqrt{k})(x-4)^k$ is ________.

(b) When $x=3$,

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}(x-4)^k=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}(-1)^k$$

Does this series converge or diverge?

(c) When $x=5$,

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}(x-4)^k=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$$

Does this series converge or diverge?

(d) The interval of convergence for the infinite series $\sum_{k=1}^{\infty}(1/\sqrt{k})(x-4)^k$ is ________.

EXERCISE SET 9.8

Graphing Utility   C CAS

1–10 Use sigma notation to write the Maclaurin series for the function.

1. e^{-x}
2. e^{ax}
3. $\cos \pi x$
4. $\sin \pi x$
5. $\ln(1+x)$
6. $\dfrac{1}{1+x}$
7. $\cosh x$
8. $\sinh x$
9. $x \sin x$
10. xe^x

11–18 Use sigma notation to write the Taylor series about $x = x_0$ for the function.

11. $e^x;\ x_0 = -1$
12. $e^{-x};\ x_0 = \ln 3$
13. $\dfrac{1}{x};\ x_0 = -1$
14. $\dfrac{1}{x+2};\ x_0 = 1$
15. $\sin \pi x;\ x_0 = \dfrac{1}{2}$
16. $\cos x;\ x_0 = \dfrac{\pi}{2}$
17. $\ln x;\ x_0 = 1$
18. $\ln x;\ x_0 = e$

19–22 Find the interval of convergence of the power series, and find a familiar function that is represented by the power series on that interval.

19. $1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$
20. $1 + x^2 + x^4 + \cdots + x^{2k} + \cdots$
21. $1 + (x-2) + (x-2)^2 + \cdots + (x-2)^k + \cdots$
22. $1 - (x+3) + (x+3)^2 - (x+3)^3 + \cdots + (-1)^k (x+3)^k + \cdots$

23. Suppose that the function f is represented by the power series
$$f(x) = 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots + (-1)^k \frac{x^k}{2^k} + \cdots$$
(a) Find the domain of f. (b) Find $f(0)$ and $f(1)$.

24. Suppose that the function f is represented by the power series
$$f(x) = 1 - \frac{x-4}{3} + \frac{(x-4)^2}{3^2} - \frac{(x-4)^3}{3^3} + \cdots$$
(a) Find the domain of f. (b) Find $f(3)$ and $f(6)$.

25–28 True–False Determine whether the statement is true or false. Explain your answer.

25. If a power series in x converges conditionally at $x = 3$, then the series converges if $|x| < 3$ and diverges if $|x| > 3$.
26. The ratio test is often useful to determine convergence at the endpoints of the interval of convergence of a power series.
27. The Maclaurin series for a polynomial function has radius of convergence $+\infty$.
28. The series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges if $|x| < 1$.

29–50 Find the radius of convergence and the interval of convergence.

29. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{2k+3}$
30. $\displaystyle\sum_{k=0}^{\infty} 4^k x^k$
31. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$
32. $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{2^k} x^k$
33. $\displaystyle\sum_{k=1}^{\infty} \frac{6^k}{k^2} x^k$
34. $\displaystyle\sum_{k=2}^{\infty} \frac{x^k}{\ln k}$
35. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$
36. $\displaystyle\sum_{k=0}^{\infty} \frac{(-2)^k x^{k+1}}{k+1}$
37. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{\sqrt{k}}$
38. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{(2k)!}$
39. $\displaystyle\sum_{k=0}^{\infty} \frac{100^k}{k!} x^k$
40. $\displaystyle\sum_{k=2}^{\infty} (-1)^{k+1} \frac{x^k}{k(\ln k)^2}$
41. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{1+k^3}$
42. $\displaystyle\sum_{k=0}^{\infty} \frac{(x-3)^k}{3^k}$
43. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x+1)^k}{k}$
44. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(x-4)^k}{(k+1)^2}$
45. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k (x+5)^k$
46. $\displaystyle\sum_{k=1}^{\infty} \frac{(2k+1)!}{k^4} (x-5)^k$
47. $\displaystyle\sum_{k=0}^{\infty} \frac{\pi^k (x-1)^{2k}}{(2k+1)!}$
48. $\displaystyle\sum_{k=0}^{\infty} \frac{(2x-1)^k}{3^{2k}}$
49. Use the root test to find the interval of convergence of
$$\sum_{k=2}^{\infty} \frac{x^k}{(\ln k)^k}$$
50. Find the domain of the function
$$f(x) = \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{(2k-2)!} x^k$$
51. Show that the series
$$1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots$$
is the Maclaurin series for the function
$$f(x) = \begin{cases} \cos \sqrt{x}, & x \geq 0 \\ \cosh \sqrt{-x}, & x < 0 \end{cases}$$
[*Hint:* Use the Maclaurin series for $\cos x$ and $\cosh x$ to obtain series for $\cos \sqrt{x}$, where $x \geq 0$, and $\cosh \sqrt{-x}$, where $x \leq 0$.]

FOCUS ON CONCEPTS

52. If a function f is represented by a power series on an interval, then the graphs of the partial sums can be used as approximations to the graph of f.
(a) Use a graphing utility to generate the graph of $1/(1-x)$ together with the graphs of the first four partial sums of its Maclaurin series over the interval $(-1, 1)$.
(b) In general terms, where are the graphs of the partial sums the most accurate?

percentage does a person's weight change in going from mean sea level to the top of Mt. Everest (29,028 ft)?

45. (a) Show that the Bessel function $J_0(x)$ given by Formula (4) of Section 9.8 satisfies the differential equation $xy'' + y' + xy = 0$. (This is called the ***Bessel equation of order zero***.)
(b) Show that the Bessel function $J_1(x)$ given by Formula (5) of Section 9.8 satisfies the differential equation $x^2y'' + xy' + (x^2 - 1)y = 0$. (This is called the ***Bessel equation of order one***.)
(c) Show that $J_0'(x) = -J_1(x)$.

46. Prove: If the power series $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ have the same sum on an interval $(-r, r)$, then $a_k = b_k$ for all values of k.

47. **Writing** Evaluate the limit

$$\lim_{x\to 0} \frac{x - \sin x}{x^3}$$

in two ways: using L'Hôpital's rule and by replacing $\sin x$ by its Maclaurin series. Discuss how the use of a series can give qualitative information about how the value of an indeterminate limit is approached.

✔QUICK CHECK ANSWERS 9.10

1. $(-1)^k \dfrac{x^{2k}}{k!}$ 2. $1; -1; 1; -1; (-1)^k x^k$ 3. $1; \dfrac{3}{2}; \dfrac{4}{3}$ 4. (a) $-\dfrac{1}{2}$ (b) $4; 1; -\dfrac{1}{4}; \dfrac{1}{18}; 4; (-1)^{k+1}\dfrac{(x-1)^k}{k\cdot(k!)}$

CHAPTER 9 REVIEW EXERCISES

1. What is the difference between an infinite sequence and an infinite series?
2. What is meant by the sum of an infinite series?
3. (a) What is a geometric series? Give some examples of convergent and divergent geometric series.
(b) What is a p-series? Give some examples of convergent and divergent p-series.
4. State conditions under which an alternating series is guaranteed to converge.
5. (a) What does it mean to say that an infinite series converges absolutely?
(b) What relationship exists between convergence and absolute convergence of an infinite series?
6. State the Remainder Estimation Theorem, and describe some of its uses.
7. If a power series in $x - x_0$ has radius of convergence R, what can you say about the set of x-values at which the series converges?
8. (a) Write down the formula for the Maclaurin series for f in sigma notation.
(b) Write down the formula for the Taylor series for f about $x = x_0$ in sigma notation.
9. Are the following statements true or false? If true, state a theorem to justify your conclusion; if false, then give a counterexample.
(a) If $\sum u_k$ converges, then $u_k \to 0$ as $k \to +\infty$.
(b) If $u_k \to 0$ as $k \to +\infty$, then $\sum u_k$ converges.
(c) If $f(n) = a_n$ for $n = 1, 2, 3, \ldots$, and if $a_n \to L$ as $n \to +\infty$, then $f(x) \to L$ as $x \to +\infty$.
(d) If $f(n) = a_n$ for $n = 1, 2, 3, \ldots$, and if $f(x) \to L$ as $x \to +\infty$, then $a_n \to L$ as $n \to +\infty$.
(e) If $0 < a_n < 1$, then $\{a_n\}$ converges.
(f) If $0 < u_k < 1$, then $\sum u_k$ converges.
(g) If $\sum u_k$ and $\sum v_k$ converge, then $\sum(u_k + v_k)$ diverges.
(h) If $\sum u_k$ and $\sum v_k$ diverge, then $\sum(u_k - v_k)$ converges.
(i) If $0 \le u_k \le v_k$ and $\sum v_k$ converges, then $\sum u_k$ converges.
(j) If $0 \le u_k \le v_k$ and $\sum u_k$ diverges, then $\sum v_k$ diverges.
(k) If an infinite series converges, then it converges absolutely.
(l) If an infinite series diverges absolutely, then it diverges.
10. State whether each of the following is true or false. Justify your answers.
(a) The function $f(x) = x^{1/3}$ has a Maclaurin series.
(b) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots = 1$
(c) $1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \cdots = 1$
11. Find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so find its limit.
(a) $\dfrac{3}{2^2 - 1^2}, \dfrac{4}{3^2 - 2^2}, \dfrac{5}{4^2 - 3^2}, \ldots$
(b) $\dfrac{1}{3}, -\dfrac{2}{5}, \dfrac{3}{7}, -\dfrac{4}{9}, \ldots$
12. Suppose that the sequence $\{a_k\}$ is defined recursively by

$$a_0 = c, \quad a_{k+1} = \sqrt{a_k}$$

Assuming that the sequence converges, find its limit if
(a) $c = \frac{1}{2}$ (b) $c = \frac{3}{2}$.
13. Show that the sequence is eventually strictly monotone.
(a) $\{(n - 10)^4\}_{n=0}^{+\infty}$ (b) $\left\{\dfrac{100^n}{(2n)!(n!)}\right\}_{n=1}^{+\infty}$
14. (a) Give an example of a bounded sequence that diverges.
(b) Give an example of a monotonic sequence that diverges.

15–20 Use any method to determine whether the series converge.

15. (a) $\sum_{k=1}^{\infty} \frac{1}{4^k}$ (b) $\sum_{k=1}^{\infty} \frac{1}{4^k+1}$

16. (a) $\sum_{k=1}^{\infty} (-1)^k \frac{2k+5}{k^2+k}$ (b) $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{k+2}{3k-1}\right)^k$

17. (a) $\sum_{k=1}^{\infty} \frac{1}{k^3+2k+1}$ (b) $\sum_{k=1}^{\infty} \frac{1}{(3+k)^{3/5}}$

18. (a) $\sum_{k=1}^{\infty} \frac{\ln k}{k\sqrt{k}}$ (b) $\sum_{k=1}^{\infty} \frac{k^{5/3}}{8k^2+5k+1}$

19. (a) $\sum_{k=1}^{\infty} \frac{9}{\sqrt{k}+1}$ (b) $\sum_{k=1}^{\infty} \frac{\cos(1/k)}{k^2}$

20. (a) $\sum_{k=1}^{\infty} \frac{k^{-1/2}}{2+\sin^2 k}$ (b) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3+1}$

21. Find the exact error that results when the sum of the geometric series $\sum_{k=0}^{\infty}(1/5)^k$ is approximated by the sum of the first 100 terms in the series.

22. Suppose: $\sum_{k=1}^{n} u_k = 3 - \frac{1}{n}$. Find

(a) u_{100} (b) $\lim_{k\to+\infty} u_k$ (c) $\sum_{k=1}^{\infty} u_k$.

23. In each part, determine whether the series converges; if so, find its sum.

(a) $\sum_{k=1}^{\infty}\left(\frac{3}{2^k} - \frac{2}{3^k}\right)$ (b) $\sum_{k=1}^{\infty}[\ln(k+1) - \ln k]$

(c) $\sum_{k=1}^{\infty} \frac{1}{k(k+3)}$ (d) $\sum_{k=1}^{\infty}[\tan^{-1}(k+1) - \tan^{-1} k]$

24. It can be proved that

$$\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

In each part, use these limits and the root test to determine whether the series converges.

(a) $\sum_{k=0}^{\infty} \frac{2^k}{k!}$ (b) $\sum_{k=0}^{\infty} \frac{k^k}{k!}$

25. Let a, b, and p be positive constants. For which values of p does the series $\sum_{k=1}^{\infty} \frac{1}{(a+bk)^p}$ converge?

26. Find the interval of convergence of

$$\sum_{k=0}^{\infty} \frac{(x-x_0)^k}{b^k} \quad (b > 0)$$

27. (a) Show that $k^k \geq k!$.

(b) Use the comparison test to show that $\sum_{k=1}^{\infty} k^{-k}$ converges.

(c) Use the root test to show that the series converges.

28. Does the series $1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \frac{5}{9} + \cdots$ converge? Justify your answer.

29. (a) Find the first five Maclaurin polynomials of the function $p(x) = 1 - 6x + 5x^2 + 3x^3$.

(b) Make a general statement about the Maclaurin polynomials of a polynomial of degree n.

30. Show that the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

is accurate to four decimal places if $0 \leq x \leq \pi/4$.

31. Use a Maclaurin series and properties of alternating series to show that $|\ln(1+x) - x| \leq x^2/2$ if $0 < x < 1$.

32. Use Maclaurin series to approximate the integral

$$\int_0^1 \frac{1-\cos x}{x}\,dx$$

to three decimal-place accuracy.

33. In parts (a)–(d), find the sum of the series by associating it with some Maclaurin series.

(a) $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

(b) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$

(c) $1 - \frac{e^2}{2!} + \frac{e^4}{4!} - \frac{e^6}{6!} + \cdots$

(d) $1 - \ln 3 + \frac{(\ln 3)^2}{2!} - \frac{(\ln 3)^3}{3!} + \cdots$

34. In each part, write out the first four terms of the series, and then find the radius of convergence.

(a) $\sum_{k=1}^{\infty} \frac{1\cdot 2\cdot 3\cdots k}{1\cdot 4\cdot 7\cdots(3k-2)} x^k$

(b) $\sum_{k=1}^{\infty} (-1)^k \frac{1\cdot 2\cdot 3\cdots k}{1\cdot 3\cdot 5\cdots(2k-1)} x^{2k+1}$

35. Use an appropriate Taylor series for $\sqrt[3]{x}$ to approximate $\sqrt[3]{28}$ to three decimal-place accuracy, and check your answer by comparing it to that produced directly by your calculating utility.

36. Differentiate the Maclaurin series for xe^x and use the result to show that

$$\sum_{k=0}^{\infty} \frac{k+1}{k!} = 2e$$

37. Use the supplied Maclaurin series for $\sin x$ and $\cos x$ to find the first four nonzero terms of the Maclaurin series for the given functions.

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

(a) $\sin x \cos x$ (b) $\frac{1}{2}\sin 2x$